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A TWO-GRID FINITE ELEMENT APPROXIMATION FOR NONLINEAR TIME FRACTIONAL TWO-TERM MIXED SUB-DIFFUSION AND DIFFUSION WAVE EQUATIONS*

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Abstract

In this paper, we develop a two-grid method (TGM) based on the FEM for 2D nonlinear time fractional two-term mixed sub-diffusion and diffusion wave equations. A two-grid algorithm is proposed for solving the nonlinear system, which consists of two steps: a nonlinear FE system is solved on a coarse grid, then the linearized FE system is solved on the fine grid by Newton iteration based on the coarse solution. The fully discrete numerical approximation is analyzed, where the Galerkin finite element method for the space derivatives and the finite difference scheme for the time Caputo derivative with order $\alpha \in (1,2)$ and $\alpha_1 \in (0,1)$. Numerical stability and optimal error estimate $O(h^{r+1} + H^{2r+2} + \tau^{\min\{3-\alpha,2-\alpha_1\}})$ in L^2-norm are presented for two-grid scheme, where t, H and h are the time step size, coarse grid mesh size and fine grid mesh size, respectively. Finally, numerical experiments are provided to confirm our theoretical results and effectiveness of the proposed algorithm.

Mathematics subject classification: 65N30, 65M60, 26A33.
Key words: Two-grid method, Finite element method, Nonlinear time fractional mixed sub-diffusion and diffusion-wave equations, L1-CN scheme, Stability and convergence.

1. Introduction

Fractional partial differential equations (FPDEs) have been the focus of many studies due to their frequent appearance in various fields such as physics, chemistry, biology and engineering [4, 8, 14, 28]. Compared with integer-order PDEs, they are better choices for describing some phenomena or processes with diffusion, relaxation vibrations, memory, hereditary and long-range interaction in viscoelasticity, electrochemistry and fluid mechanics.

In this paper, we consider the numerical solution of the following nonlinear time-fractional two-term mixed sub-diffusion and diffusion wave equations:

$$\begin{cases} {}_0^C D_t^{\alpha_1} u(\boldsymbol{x},t) + {}_0^C D_t^{\alpha} u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = g(\boldsymbol{u}), & (\boldsymbol{x},t) \in \Omega \times (0,T], \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad u_t(\boldsymbol{x},0) = \tilde{u}_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal region with boundary $\partial\Omega$, $\boldsymbol{x}$=[x,y], $g(\cdot)$ is twice continuously differentiable.

The Caputo fractional derivative ${}_0^C D_t^{\alpha_1}$, ${}_0^C D_t^{\alpha}$ are defined by ([14])

$$ {}_0^C D_t^{\alpha_1} u(\boldsymbol{x},t) = \frac{1}{\Gamma(1-\alpha_1)} \int_0^t \frac{\partial u(\boldsymbol{x},s)}{\partial s} \frac{ds}{(t-s)^{\alpha_1}}, \qquad 0<\alpha_1<1. \tag{1.2} $$

$$ {}_0^C D_t^{\alpha} u(\boldsymbol{x},t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(\boldsymbol{x},s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \qquad 1<\alpha<2. \tag{1.3} $$

The nonlinear time-fractional two-term mixed sub-diffusion and diffusion wave equations have been widely applied in depicting the anomalous diffusion process, modeling viscoelastic damping, capturing power-law frequency dependence [6, 7, 10, 25, 27]. There have existed some jobs in the area of numerical analysis for linear fractional mixed sub-diffusion and diffusion wave equations. Sun [20] proposed a new analytical technique of the L-type difference schemes for time fractional mixed sub-diffusion and diffusion wave equations with the $\min\{2-\alpha_1, 3-\alpha\}$ order accuracy in a discrete H^1-norm in time and the second order accuracy in space, respectively. A Galerkin finite element method combined with L1-CN time discrete scheme for finding the numerical solution of two-term time-fractional mixed diffusion and diffusion wave equations in [22]. By use of anisotropic linear triangle finite element method, Zhao [27] presented a fully-discrete scheme for multi-term time-fractional mixed diffusion and diffusion wave equations with variable coefficient on 2D bounded domain. To the best of our knowledge, no article is available in the literature concerning a numerical analysis for fully discrete finite element approximations for the nonlinear time-fractional two-term mixed sub-diffusion and diffusion wave equations.

As we all know, the TGM is usually regarded as an efficient discretization technique for solving the nonsymmetric indefinite and nonlinear equations based on a coarse mesh with size H and a finer mesh with size h ($h \ll H$). More precisely, a nonlinear or nonsymmetric problem is solved on the coarse mesh. Then the solution obtained from coarse grid is used as a initial guess to solve a linearized problem on the finer mesh [23, 24]. Later on, Chen [1, 2] proposed two grid mixed finite element methods to solve nonlinear reaction-diffusion equations. Liu [15, 16] considered a two-grid finite element approximation for a nonlinear time-fractional Cable equation and nonlinear fourth-order fractional differential equations with Caputo fractional derivative. Recently, Chen [11,12] also have done some work on the two grid method of fractional differential equations.

In this article, our main task is to take the Galerkin finite element to construct a fully discrete TGM scheme for 2D nonlinear time fractional two-term mixed sub-diffusion and diffusion wave equations, and derive the analysis of the corresponding stability and the error estimate in L^2-norm.

The remainder of the paper is organized as follows. In Section 2, we propose a fully-discrete scheme for (1.1) based on Galerkin FEM and L1-CN approximation. The unconditional stability analysis and the corresponding error estimate are deduced. In Section 3, we set up the TGM, and the stability and a priori error estimate of TGM are proved. In Section 4, the numerical example is presented to verify our theoretical analysis and some comparisons of computing time are done. The paper is concluded with some remarks in the last section.

Throughout this paper, let $L^p(\Omega)$ be the Lebesgue space with norm $\|\cdot\|_{0,p}$ for $0 \le p \le \infty$,