A STOCHASTIC ALGORITHM FOR FAULT INVERSE PROBLEMS IN ELASTIC HALF SPACE WITH PROOF OF CONVERGENCE[*]

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Abstract

A general stochastic algorithm for solving mixed linear and nonlinear problems was introduced in [11]. We show in this paper how it can be used to solve the fault inverse problem, where a planar fault in elastic half-space and a slip on that fault have to be reconstructed from noisy surface displacement measurements. With the parameter giving the plane containing the fault denoted by $\boldsymbol{m}$ and the regularization parameter for the linear part of the inverse problem denoted by C, both modeled as random variables, we derive a formula for the posterior marginal of $\boldsymbol{m}$. Modeling C as a random variable allows to sweep through a wide range of possible values which was shown to be superior to selecting a fixed value [11]. We prove that this posterior marginal of $\boldsymbol{m}$ is convergent as the number of measurement points and the dimension of the space for discretizing slips increase. Simply put, our proof only assumes that the regularized discrete error functional for processing measurements relates to an order 1 quadrature rule and that the union of the finite-dimensional spaces for discretizing slips is dense. Our proof relies on trace class operator theory to show that an adequate sequence of determinants is uniformly bounded. We also explain how our proof can be extended to a whole class of inverse problems, as long as some basic requirements are met. Finally, we show numerical simulations that illustrate the numerical convergence of our algorithm.

Mathematics subject classification: 45Q05, 65J20, 60F99.
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1. Introduction

In [11], we introduced a numerical method for mixed linear and nonlinear inverse problems. This method applies to cases where the data for the inverse problem is corrupted by noise and where for each value of the nonlinear parameter, the underlying linear problem is ill-posed. Accordingly, regularizing this linear part is required. The norm used for the regularization process has to be multiplied by a scaling parameter, also called regularization parameter, denoted by C throughout this paper. In [11], a Bayesian approach was adopted, and C was modeled as a random variable. It was shown in [11] that this approach is superior to selecting C using some standard method for linear inverse problems, such as the discrepancy principle, or the generalized cross validation. Loosely speaking, this can be explained by observing that for different values of the nonlinear parameter $\boldsymbol{m}$, these classical methods will favor different values

of C, and as a result different values of the nonlinear parameter $\boldsymbol{m}$ cannot be fairly compared. Attempting to select a unique value of C for all values of $\boldsymbol{m}$ leads to somehow better results, but as demonstrated in the last section of this paper, doing so pales in comparison to the method advocated in [11].

In this paper, we first derive in Section 3 a specific version of the Bayesian posterior distribution of $\boldsymbol{m}$ following the method introduced in [11]. This version applies to an inverse problem in half space for the linear elasticity equations. In the direct problem, a slip field on an open surface that we will call a fault, gives rise to a displacement field. In the inverse problem, this field is measured on the plane on top of the half space at M points. The linear part of the inverse problem consists of reconstructing the slip field on the fault. The nonlinear part consists of finding the geometry and the location of the fault. This formulation is commonly used in geophysics to model slow slip events in the vicinity of subduction zones, or the total displacement resulting from a dynamic earthquake, see [13, 14] and references therein. In this paper, the geometry of the fault is assumed to be planar, thus we choose the nonlinear parameter $\boldsymbol{m}$ to be in $\mathbb{R}^3$. The stochastic model considered in this paper is different from a model studied in an earlier paper by the same author [12]. As explained in section 3, the difference is that we here assume the regularization parameter C for the linear part of the inverse problem to be a random variable and the covariance σ of the measurements is unknown. In section 3, the likelihood of σ is optimized based on the data and as a consequence the resulting posterior distribution function of $\boldsymbol{m}$ is entirely different from the simpler one used in [12]. Computing this new posterior is more intricate since it involves the random variable $(\boldsymbol{m}, C)$ instead of just $\boldsymbol{m}$ in [12]. The benefit of using this new posterior is that it leads to much better numerical results if the covariance σ of the measurements is unknown as shown in sections 6.2 and 6.3.

In Section 4, we provide a mathematical proof of the soundness of our Bayesian approach for computing the posterior probability density of $\boldsymbol{m}$. More precisely, we show that as the number of measurement points M tends to infinity and the dimension of the space for discretizing slip fields tends to infinity, the probability of $\boldsymbol{m}$ to be further than a fixed $\eta > 0$ from the true value $\tilde{\boldsymbol{m}}$ converges to zero if the noise level is low enough. Interestingly, although the derivation of the probability law of the posterior of $\boldsymbol{m}$ assumes that the noise is Gaussian, once this law is set the proof of convergence does not require the noise to be Gaussian. The proof assumes that the regularized discrete error functional for sampling measurements relates to an order 1 quadrature rule for C^1 functions (which is verified by most commonly used quadrature rules) and that the union of the finite-dimensional spaces for discretizing slips is dense, a natural requirement that can be easily satisfied by combining adequate finite element spaces. The more difficult step in this proof of convergence requires showing that a sequence of determinants is uniformly bounded: this can be achieved by using trace class operator theory [6].

In Section 5, for the reader's convenience, we extend the formulation of our recovery method to more general inverse problems. We write down the expression of the posterior probability distribution that is at the core of our reconstruction method. A companion parallel sampling algorithm is given in [11], Section 3.2. At the end of Section 5 of this paper, we state a convergence theorem for the posterior probability distribution which is valid in this more general framework.

In Section 6, we present numerical simulations that use our posterior probability distribution for reconstructing the geometry parameter $\boldsymbol{m}$. The posterior marginals of $\boldsymbol{m}$ and of the regularization parameter C are computed using the method of choice sampling due to the size of the search space. We used a modified version of the Metropolis algorithm which is well suited