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SECOND ORDER UNCONDITIONALLY STABLE AND CONVERGENT LINEARIZED SCHEME FOR A FLUID-FLUID INTERACTION MODEL*

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Abstract

In this paper, a fully discrete finite element scheme with second-order temporal accuracy is proposed for a fluid-fluid interaction model, which consists of two Navier-Stokes equations coupled by a linear interface condition. The proposed fully discrete scheme is a combination of a mixed finite element approximation for spatial discretization, the second-order backward differentiation formula for temporal discretization, the second-order Gear's extrapolation approach for the interface terms and extrapolated treatments in linearization for the nonlinear terms. Moreover, the unconditional stability is established by rigorous analysis and error estimate for the fully discrete scheme is also derived. Finally, some numerical experiments are carried out to verify the theoretical results and illustrate the accuracy and efficiency of the proposed scheme.

Mathematics subject classification: 65M15, 65M60.
Key words: Fluid-fluid interaction model, Unconditional stability, Second order temporal accuracy, Error estimate.

1. Introduction

Numerical simulation of multi-domain and multi-physics coupling of one fluid with another fluid is an important aspect in many industrial applications. In fact, the fluid-fluid interaction model can be seen as one of them arises in many important scientific, engineering and industrial applications, such as heterogeneous of blood flow [8] and atmosphere-ocean interaction [20–22]. Due to the practical importance of the fluid-fluid interaction problem, there has been a lot of attention recently paid to the development of accurate and efficient numerical methods; see, e.g., [5,16–19,23] among many others. Besides, Bresch and Koko [4] have presented a numerical simulation of the considered model by using an operator-splitting method and optimization-based nonoverlapping domain decomposition methods. Based on implicit-explicit scheme for the nonlinear interface conditions, Connors et al. [7] have presented a decoupled time stepping method, which is conditionally stable proved by Zhang et al. [25]. Recently, Aggul et al. [2] have developed a predictor-corrector-type method that is an unconditionally stable scheme with second order time accuracy.

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In this paper, we study the following governing equations of a fluid-fluid interaction model [9, 26]. Let a bounded domain $\Omega \subset \mathbb{R}^2$ consist of two sub-domains Ω_1 and Ω_2 coupled across their shared interface I, for times $t \in [0, T]$. For $i = 1, 2$, given the kinematic viscosities $\nu_i > 0$, the friction coefficients $\kappa > 0$, the body forces $f_i : [0, T] \to H^1(\Omega_i)^2$, and initial values $u_{i,0} \in H^1(\Omega_i)^2$, find the fluid velocities $u_i : [0, T] \times \Omega_i \to \mathbb{R}^2$ and pressures $p_i : [0, T] \times \Omega_i \to \mathbb{R}$ satisfying (for $t \in (0, T]$)

$$\begin{aligned}
&u_{i,t} - \nu_i \Delta u_i + u_i \cdot \nabla u_i + \nabla p_i = f_i && \text{in } \Omega_i, \\
&-\nu_i n_i \cdot \nabla u_i \cdot \tau = \kappa(u_i - u_j) \cdot \tau && \text{on } I, \text{ for } i, j = 1, 2, \text{ and } i \neq j, \\
&u_i \cdot n_i = 0 && \text{on } I, \\
&\nabla \cdot u_i = 0 && \text{in } \Omega_i, \\
&u_i(0, x) = u_{i,0}(x) && \text{in } \Omega_i, \\
&u_i = 0 && \text{on } \Gamma_i := \partial\Omega_i \backslash I.
\end{aligned} \tag{1.1}$$

The vectors n_i are the unit normals on $\partial\Omega_i$, and τ is any vector on I such that $\tau \cdot n_i = 0$. Note that the linear interface conditions are considered on the interface I, which have been studied in past score years. Lions et al. [22] and Friedlander and Serre [9] have proved the existence, uniqueness and regularity of the solution of the problem (1.1). Recently, Zhang et al. [26] have proved that the error estimates of a decoupled scheme for the velocities in H^1 norm and pressures in L^2 norm are $\Delta t^{\frac{7}{8}} + h$ and $\Delta t^{\frac{3}{4}} + h$, respectively. However, the decoupled scheme is conditionally convergent with $\Delta t \leq ch^{\frac{1}{2}}$. Besides, for the same interface condition as problem (1.1), Connors et al. [6] have proposed a partitioned time stepping method for a parabolic two-domain problem and analyzed the error estimates.

In this paper, the purpose of the current efforts is to propose and investigate a fully discrete finite element scheme with second order temporal accuracy for the fluid-fluid interaction model (1.1). We discretize the system in time via a combination of second order backward differentiation formula (BDF) for the temporal terms, second order Gear's extrapolation approach for the interface terms and extrapolated treatments in linearization for the nonlinear terms. The coupling terms in the interface conditions are treated explicitly in our scheme so that only two decoupled Navier-Stokes equations are solved at each time step.

The rest of the paper is arranged as follows: In the next section, we introduce some mathematical preliminaries and provide the corresponding variational form for the problem (1.1). In Section 3, we propose a fully discrete finite element scheme for the fluid-fluid interaction model. Besides, the unconditional stability of the presented scheme is proven. Then in Section 4, we derive and prove the error estimates for the considered scheme. In Section 5, some numerical experiments are implemented to verify the theoretical results and efficiency of the proposed scheme. Consequently, we end our paper by drawing a conclusion in the last section.

2. Notation and Preliminaries

In this section, we describe some necessary definitions and inequalities, which will be frequently applied to the following sections. We introduce the usual $L^2(\Omega_i)$ norm and its inner product by $\|\cdot\|_0$ and $(\cdot,\cdot)_{\Omega_i}$, respectively. The $L^p(\Omega_i)$ norms and the Sobolev $W_p^m(\Omega_i)$ norms are denoted by $\|\cdot\|_{L^p(\Omega_i)}$ and $\|\cdot\|_{W_p^m(\Omega_i)}$ for $m \in \mathbb{N}^+$, $1 \leq p \leq \infty$. In particular, $H^m(\Omega_i)$ is used to represent the Sobolev space $W_2^m(\Omega_i)$ and $\|\cdot\|_m$ denotes the norm in $H^m(\Omega_i)$. For