Journal of Computational Mathematics
Vol.41, No.1, 2023, 94–106.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2104-m2020-0233

UNCONDITIONAL SUPERCONVERGENT ANALYSIS OF QUASI-WILSON ELEMENT FOR BENJAMIN-BONA-MAHONEY EQUATION*

Xiangyu Shi[1)]
Business School, Zhengzhou University, Zhengzhou 450001, China;
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email: 19020170155426@stu.xmu.edu.cn
Linzhang Lu
School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, China;
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email: llz@gznu.edu.cn, lzlu@xmu.edu.cn

Abstract

This article aims to study the unconditional superconvergent behavior of nonconforming quadrilateral quasi-Wilson element for nonlinear *Benjamin Bona Mahoney* (BBM) equation. For the generalized rectangular meshes including rectangular mesh, deformed rectangular mesh and piecewise deformed rectangular mesh, by use of the special character of this element, that is, the conforming part(bilinear element) has high accuracy estimates on the generalized rectangular meshes and the consistency error can reach order $O(h^2)$, one order higher than its interpolation error, the superconvergent estimates with respect to mesh size h are obtained in the broken H^1-norm for the semi-/ fully-discrete schemes. A striking ingredient is that the restrictions between mesh size h and time step τ required in the previous works are removed. Finally, some numerical results are provided to confirm the theoretical analysis.

Mathematics subject classification: 65N15, 65N30.
Key words: BBM equations, Quasi-Wilson element, Superconvergent behavior, Semi-and fully-discrete schemes, Unconditionally.

1. Introduction

In this paper, we consider the following nonlinear BBM equation:

$$\begin{cases} u_t - \Delta u_t = \nabla \cdot \vec{f}(u), & (X,t) \in \Omega \times (0,T], \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega. \end{cases} \tag{1.1}$$

Where $0 < T < \infty$, $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with the boundary $\partial\Omega$, $X = (x,y)$, $u_t = \frac{\partial u}{\partial t}$, $u_0(X)$ is a known sufficiently smooth function and $\vec{f}(u) = (-(\frac{1}{2}u^2 + u), -(\frac{1}{2}u^2 + u))$.

As we know, there have been some studies about the theoretical analysis and numerical simulation of finite element methods (FEMs) for problem (1.1). For example, the convergence of conforming *Crank-Nicolson* (CN) fully-discrete Galerkin FEM was discussed in [1]. The superconvergence of Galerkin FEMs for conforming element and nonconforming rectangular

* Received September 2, 2020 / Revised version received January 3, 2021 / Accepted April 29, 2021 / Published online March 27, 2022 /
1) Corresponding author

EQ_1^{rot} element (see [4]) were studied in [2] and [3], respectively. Recently, the superconvergent analysis of an H^1-Galerkin FEM with conforming element pair was presented in [5]. A new mixed FEM and its' superconvergent behavior with nonconforming constrained rotated Q_1 element and constant pair was developed in [6]. The two-grid method for BDF2 scheme with bilinear element was investigated in [7]. The main advantage of [6] and [7] is that there is no restriction between h and τ.

On the other hand, it has been proven in [8] that the consistency error of the famous rectangular Wilson element is of order $O(h)$ and cannot be improved anymore even the exact solution is smooth enough. It has been shown in [9] that the consistency errors of quadrilateral quasi-Wilson elements of [10] are of order $O(h^2)$. Later on, these elements and their modified forms of [11, 12] have been widely applied to some PDEs for superconvergent analysis (see [13–16]). But up to now, there is no report on the application to BBM equation.

In the present work, we will attempt to use the quasi-Wilson element of [9] to solve problem (1.1). Then, for generalized quadrilateral meshes including rectangular mesh, deformed rectangular mesh and piecewise deformed rectangular mesh(see [17, 18]), we derive the superconvergent estimates/ unconditional superconvergent estimates for the semi-discrete scheme/ the Backward Euler (BE) and CN schemes on quadrilateral meshes by proving the boundedness of the numerical solution in the broken H^1-norm instead of L^∞-norm, which improves the results of [2, 3].

The rest of this paper is organized as follows: In section 2, some important estimates of quasi-Wilson element are introduced. In section 3, the superclose estimate with order $O(h^2)$ for the semi-discrete scheme is derived. In sections 4-5, the superclose estimates are obtained for both BE and CN fully-discrete schemes with order $O(h^2+\tau)$ and $O(h^2+\tau^2)$ without the restriction between h and τ, respectively. In section 6, the unconditional global superconvergent results of the above three schemes are gained through interpolated post-processing technique. In the last section, some numerical results are given to show the performance of our method.

2. Some Estimates of Quasi-Wilson Element

Let $\hat{K}=[-1,1]^2$ be the reference element on $\xi-\eta$ plane with four vertices $\hat{A}_1=(-1,-1)$, $\hat{A}_2=(1,-1)$, $\hat{A}_3=(1,1)$ and $\hat{A}_4=(-1,1)$. We define the quasi-Wilson element $\{\hat{K},\hat{P},\hat{\Sigma}\}$ on $\hat{K}$ as [9, 15]:

$$\hat{P}=span\{N_i(\xi,\eta)\ (i=1,2,3,4),\ \hat{\psi}(\xi),\ \hat{\psi}(\eta)\},$$
$$\hat{\Sigma}=\{\hat{v}(\hat{a}_i),i=1,2,3,4;\frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi^2}d\xi d\eta,\frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\eta^2}d\xi d\eta\}.$$

where $N_i(\xi,\eta)=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$, $(\xi_1,\xi_2,\xi_3,\xi_4)=(-1,1,1,-1)$, $(\eta_1,\eta_2,\eta_3,\eta_4)=(-1,-1,1,1)$, $\hat{\psi}(s)=\frac{1}{2}(s^2-1)-\frac{5}{12}(s^4-1)$ and $\hat{v}_i=\hat{v}(\hat{A}_i),\ i=1,2,3,4$.

Obviously, the only difference between this element and the classical Wilson element is the change of $\psi(\cdot)$. Let T_h be a family of regular convex quadrilateral subdivision of Ω, $K\in T_h$ be an element with vertices $A_i(x_i,y_i),\ 1\le i\le 4$, then there exists a mapping F_K given by

$$x^K=\sum N_i(\xi,\eta)x_i,\qquad y^K=\sum N_i(\xi,\eta)y_i,$$

such that

$$F_K(\hat{A}_i)=A_i,\qquad F_K(\hat{K})=K.$$