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EXPONENTIAL TIKHONOV REGULARIZATION METHOD FOR SOLVING AN INVERSE SOURCE PROBLEM OF TIME FRACTIONAL DIFFUSION EQUATION*

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Abstract

In this paper, we mainly study an inverse source problem of time fractional diffusion equation in a bounded domain with an over-specified terminal condition at a fixed time. A novel regularization method, which we call the exponential Tikhonov regularization method with a parameter γ, is proposed to solve the inverse source problem, and the corresponding convergence analysis is given under a-priori and a-posteriori regularization parameter choice rules. When γ is less than or equal to zero, the optimal convergence rate can be achieved and it is independent of the value of γ. However, when γ is great than zero, the optimal convergence rate depends on the value of γ which is related to the regularity of the unknown source. Finally, numerical experiments are conducted for showing the effectiveness of the proposed exponential regularization method.

Mathematics subject classification: 35R30, 35R11, 65M32.
Key words: Exponential regularization method, Inverse source problem, Fractional diffusion equation, Ill-posed problem, Convergence rate.

1. Introduction

In the last few decades, the fractional diffusion equations have received great attention by many researchers due to their potential applications for simulating real physical phenomena. For example, the fractional diffusion equations can be used to model the relaxation phenomena in complex viscoelastic materials [1], the anomalous transport in underground environmental experiments [2], the anomalous diffusion in the laboratory-scale heterogeneous porous media [3]. However, in many practical applications, the source term, initial distribution, a part of boundary

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distribution, diffusion coefficients or the order of fractional derivative might be not available, and they need to be reconstructed from extra measurement data, which generate a variety of inverse problems of fractional diffusion equations [4–18]. Here, we consider an inverse problem of time fractional diffusion equation for reconstructing an unknown source.

Let Ω be a bounded domain in R^d $(1 \leq d \leq 3)$ with smooth boundary $\partial\Omega$, and $L^2(\Omega)$ be the square-integrable function space with the scalar product $(\cdot,\cdot)$ defined by $(f(x), h(x)) = \int_\Omega f(x)h(x)dx$. Consider a homogeneous initial-boundary value problem of time fractional diffusion equation as follows:

$$\begin{cases} {}_0\partial_t^\alpha u(x,t) = (Lu)(x,t) + f(x), & x \in \Omega, \quad t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = 0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where ${}_0\partial_t^\alpha u$ is the Caputo time-fractional derivative defined by

$${}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u(x,s)}{\partial s} ds, \quad 0 < \alpha < 1, \tag{1.2}$$

L is a linear elliptic operator of second order on $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$, i.e.,

$$Lu = \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x)u(x), \quad x \in \Omega. \tag{1.3}$$

Moreover, the elliptic operator L is assumed to be symmetric and uniform with $a_{ij}(x) \in C^1(\bar{\Omega}), c(x) \in C(\bar{\Omega})$, which means that

$$\begin{aligned} &a_{ij}(x) = a_{ji}(x), \qquad 1 \leq i,j \leq d, \\ &0 < \theta \sum_{i=1}^d |\xi_i|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq \eta \sum_{i=1}^d |\xi_i|^2, \qquad \xi \in R^d, \ x \in \bar{\Omega} \end{aligned}$$

with positive constants θ and η.

The inverse problem considered in this paper is to reconstruct the unknown source $f(x)$ from system (1.1) and the following terminal data

$$u(x,T) = g^\delta(x), \tag{1.4}$$

where $g^\delta(x) \in L^2(\Omega)$ is the measurement data of the exact value $g(x)$ with

$$\left\| g^\delta(x) - g(x) \right\| \leq \delta. \tag{1.5}$$

Here, δ is the noise level, and $\| \cdot \|$ is the L^2-norm that is induced by the scalar product in $L^2(\Omega)$.

As we all know, the above inverse source problem is ill-posed in the sense of Hadamard, while the direct problem is well-posed [19]. Therefore some regularized techniques must be adopted to deal with this inverse source problem. Especially, if $\alpha = 1$, the problem is an inverse source problem of parabolic equation, which has been studied widely by many researchers [20–23]. Moreover, the regularization method of the inverse heat source problem is used to treat the problem of numerical differentiation [21]. For the time fractional diffusion equations, i.e. the