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A VARIATIONAL ANALYSIS FOR THE MOVING FINITE ELEMENT METHOD FOR GRADIENT FLOWS*

Xianmin Xu
LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, NCMIS, AMSS, Chinese Academy of Sciences, Beijing 100190, China;
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
Email: xmxu@lsec.cc.ac.cn

Abstract

By using the Onsager principle as an approximation tool, we give a novel derivation for the moving finite element method for gradient flow equations. We show that the discretized problem has the same energy dissipation structure as the continuous one. This enables us to do numerical analysis for the stationary solution of a nonlinear reaction diffusion equation using the approximation theory of free-knot piecewise polynomials. We show that under certain conditions the solution obtained by the moving finite element method converges to a local minimizer of the total energy when time goes to infinity. The global minimizer, once it is detected by the discrete scheme, approximates the continuous stationary solution in optimal order. Numerical examples for a linear diffusion equation and a nonlinear Allen-Cahn equation are given to verify the analytical results.

Mathematics subject classification: 65M12, 65M55.
Key words: Moving finite element method, Convergence analysis, Onsager principle.

1. Introduction

The moving finite element method (MFEM) was first developed in [27,28] about forty years ago. It is a typical r-type adaptive method [6,7,17,22,34], where the mesh vertexes are relocated without changing the mesh topology. In the MFEM, the mesh relocation is done by solving a dynamic equation for the vertexes coupled with the original partial differential equations. No interpolation is needed in the method since the mesh is continuous with time. The MFEM has arisen considerable interest and has been further developed in several directions (c.f [1–3,8,16,36] among many others).

However, like all other r-adaptive methods, the theoretical analysis for the MFEM is far from being complete. The first error analysis for MFEM was done by Dupont [16], where he proved the optimal convergence of the method for a linear convection diffusion equation when the solution is smooth. This is not enough since we are more interested in non-smooth solutions for adaptive methods. Later on, Jimack proved the locally optimal approximation for the stationary solution of a linear parabolic equation without the smoothness assumption [19–21]. In this study, we aim to do analysis for a nonlinear gradient flow system by using the Onsager variational principle as an approximation tool. Recently, a similar energetic variational approach is used to develop interesting Lagrangian schemes for some gradient flow systems [9, 23, 24].

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The Onsager variational principle is a fundamental principle for irreversible thermodynamic processes in statistical physics [14, 29, 30]. It has be used to model many dissipative physical systems [13, 14], such as the Stokes equation in hydrodynamics, the Ericksen-Leslie equation in liquid crystal, and the GNBC boundary condition for moving contact lines [33], etc. Recently, the Onsager principle has been used as an approximation tool for many problems in two-phase flows and in material science [12, 15, 18, 26, 37, 38]. In particular, it has been used to derive an efficient numerical method for wetting dynamics [25].

In this work, we first give a new derivation of the MFEM for a gradient flow system by using the Onsager principle as an approximation tool. The key idea is to approximate the system in a nonlinear approximation space of free-knot piecewise polynomials [10]. Both the mesh vertexes and the nodal values of the finite element function are regarded as unknowns. We derive a system of ordinary differential equations(ODEs) for them. The ODE system coincides with the discrete equation of the MFEM, which has been derived in a totally different way in [28]. Here we do not need to compute the multiply of a Dirac measure and a discontinuous function, so that the "mollification" technique or any other formally calculation is not needed. Furthermore, our derivation shows that the discretized problem has the same energy dissipation structure of the continuous system. This makes us to prove the energy decay property of the discrete problem easily.

Based on the variational formula, we do error analyse for the MFEM for a stationary solution of the gradient flow system. The analysis can be regarded as a generalization of the results in [20] to nonlinear equations. We show that the MFEM gives locally best approximations to the energy. When a global minimizer is detected, an optimal error estimate is proved using the nonlinear approximation theory. Numerical examples show that the optimal convergence can be obtained for a linear diffusion equation and for stationary solutions of a nonlinear Allan-Cahn equation. In this paper, we mainly consider the one dimensional problem. All the results can be generalized to higher dimensional cases directly.

The structure of the paper is as follows. In section 2, we introduce the Onsager variational principle and show that it can be used to derive the partial differential equation model for a gradient flow system. In Section 3, we derive the MFEM by using the Onsager principle as an approximation tool. In Section 4, we do error analysis for the stationary solution of a nonlinear reaction diffusion equation. Some numerical examples are illustrated to verify the analytical results in the last section.

2. The Onsager Variational Principle for a Gradient Flow System

2.1. The Onsager principle

Suppose a physical system is described by a time dependent function u. For simplicity, we denote by $\dot{u} = \frac{\partial u}{\partial t}$ the time derivative of u. For a dissipated system, the evolution of u can dissipate energy. The dissipation function is defined as half of the total energy dissipated with respect to the flux $\dot{u}$(c.f. [14]). For a simple gradient flow system, we assume the dissipation function is

$$\Phi(\dot{u}) = \frac{\xi}{2}\|\dot{u}\|^2, \tag{2.1}$$

where ξ is a positive friction coefficient determined by the dissipation processes of a physical system and $\|\cdot\|$ is a L^2 norm.