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UNCONDITIONALLY OPTIMAL ERROR ANALYSIS OF THE SECOND-ORDER BDF FINITE ELEMENT METHOD FOR THE KURAMOTO-TSUZUKI EQUATION*

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Abstract

This paper aims to study a second-order semi-implicit BDF finite element scheme for the Kuramoto-Tsuzuki equations in two dimensional and three dimensional spaces. The proposed scheme is stable and the nonlinear term is linearized by the extrapolation technique. Moreover, we prove that the error estimate in L^2-norm is unconditionally optimal which means that there has not any restriction on the time step and the mesh size. Finally, numerical results are displayed to illustrate our theoretical analysis.

Mathematics subject classification: 65N12, 65N15, 65N30.
Key words: Kuramoto-Tsuzuki equations, BDF scheme, Finite element method, Optimal error analysis.

1. Introduction

In this paper, we will study the unconditionally optimal error estimates of the second-order BDF finite element scheme for the following Kuramoto-Tsuzuki equation:

$$u_t = (1 + ic_1)\Delta u + u - (1 + ic_2)|u|^2u, \quad \text{in } \Omega \times (0, T] \tag{1.1}$$

for some $T > 0$, where $i = \sqrt{-1}$, and $\Omega \subset \mathbf{R}^d$ is a bounded and convex polygon ($d = 2$) or polyhedron ($d = 3$) domain. In (1.1), the unknown u and the initial value u_0 are complex value functions, and c_1, c_2 are real constants. To ensure the well-posedness of the solution to the Kuramoto-Tsuzuki equation (1.1), the proper initial and boundary conditions are needed. For the sake of simplicity, we consider the following initial condition

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \tag{1.2}$$

and the homogeneous Dirichlet boundary condition

$$u = 0, \quad \text{on } \partial\Omega \times (0, T]. \tag{1.3}$$

The Kuramoto-Tsuzuki equation describes the behavior of many two-component systems in a neighborhood of the bifurcation point [7] and can be viewed as a special case of the Ginzburg-Landau equations in the theory of superconductivity [6].

It is clear that the Kuramoto-Tsuzuki equation is a nonlinear parabolic equation and the analytic solution can not be solved from (1.1)–(1.3) directly. Numerical methods for the numerical

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simulations of the Kuramoto-Tsuzuki equation have been developed by many scholars. We first describe some works for the one dimensional Kuramoto-Tsuzuki equation. Tsertsvadze in [21] proposed a nonlinear Crank-Nicolson finite difference scheme and derived the convergence rate of order $O(h^{3/2})$ in the discrete L^2-norm under the constraint $\tau = O(h^{2+\delta})$ for some $\delta > 0$, where τ and h denote the time step size and the mesh size, respectively. The second-order convergence rate of order $O(\tau^2 + h^2)$ in L^∞-norm for Tsertsvadze's difference scheme was proved by Sun without any constraint on the time step size and the mesh size [20]. Another nonlinear Crank-Nicolson finite difference scheme was studied in [25], where the optimal second-order convergence rate was proved. Based upon the linear extrapolation, a semi-implicit Crank-Nicolson finite element scheme was proposed and studied by Omrani in [16]. Other different numerical schemes for the one dimensional Kuramoto-Tsuzuki equation can be found in [5, 8, 18, 19, 24] and references cited therein.

Now we turn back to the high dimensional problem. In the last several decades, numerous effort has been devoted to the development of efficient numerical methods for the nonlinear parabolic problems which arise from a variety of physical and industrial applications. In designing numerical schemes, a key issue is the time step condition. Usually, fully implicit schemes are unconditionally stable, but one has to solve a nonlinear system by some iteration method at each time step, which results in the time consumption, especially for 3D problems. Explicit schemes are much easy in computation. But there has a very restricted time step condition for the convergence of numerical solutions. A popular and widely-used method is a linearized semi-implicit scheme, such as the first-order semi-implicit Euler scheme, and the second-order semi-implicit Crank-Nicolson or BDF scheme based upon the linear extrapolation. To study the error estimate of the semi-implicit scheme, the boundedness of numerical solutions in L^∞-norm are often required. One can use the induction method with inverse inequality to bound it, such as

$$\begin{aligned} \|U_h^n\|_{L^\infty} &\le \|R_h u^n\|_{L^\infty} + \|R_h u^n - U_h^n\|_{L^\infty} \\ &\le \|R_h u^n\|_{L^\infty} + Ch^{-d/2}\|R_h u^n - U_h^n\|_{L^2} \\ &\le \|R_h u^n\|_{L^\infty} + Ch^{-d/2}(\tau^p + h^{r+1}), \end{aligned} \tag{1.4}$$

where u^n and U_h^n are the exact solution and numerical solution, respectively, R_h is the classical Ritz projection operator, and p and $r+1$ are the convergence rates in temporal and spatial directions, respectively. The estimate (1.4) results in a time step constraint on the mesh size. We notice that such analysis method has been widely used in the error estimates of fully discrete schemes for many different nonlinear parabolic problems until [11, 12], in which the unconditionally optimal error estimate without the above time step constraint was proved by using a technique of error splitting, i.e., the error is splitted into the temporal error, the spatial error and the projection error by introducing a corresponding time-discrete parabolic system (or elliptic system). This technique has been successfully applied to some semi-implicit schemes for nonlinear parabolic problems and the unconditionally optimal error estimates in L^2-norm were proved, such as the nonlinear thermistor equation [10], the Landau-Lifshitz equation [1,3], the nonlinear Schrödinger equation [14, 22], the miscible displacement in porous media [23] and other strongly nonlinear parabolic problems [13, 15]. Besides the technique of error splitting, another way to prove the unconditionally optimal error estimate was proposed for the Crank-Nicolson difference scheme for a coupled nonlinear Schrödinger system in [17], where the optimal error estimate in the discrete L^2-norm were derived for $\tau \le h$ and $\tau > h$, respectively, then the unconditionally optimal convergence rate was proved. Based upon this approach in [17],