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SUPERCONVERGENCE ANALYSIS OF A BDF-GALERKIN FEM FOR THE NONLINEAR KLEIN-GORDON-SCHRÖDINGER EQUATIONS WITH DAMPING MECHANISM*

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Abstract

The focus of this paper is on a linearized backward differential formula (BDF) scheme with Galerkin FEM for the nonlinear Klein-Gordon-Schrödinger equations (KGSEs) with damping mechanism. Optimal error estimates and superconvergence results are proved without any time-step restriction condition for the proposed scheme. The proof consists of three ingredients. First, a temporal-spatial error splitting argument is employed to bound the numerical solution in certain strong norms. Second, optimal error estimates are derived through a novel splitting technique to deal with the time derivative and some sharp estimates to cope with the nonlinear terms. Third, by virtue of the relationship between the Ritz projection and the interpolation, as well as a so-called "lifting" technique, the superconvergence behavior of order $O(h^2+\tau^2)$ in H^1-norm for the original variables are deduced. Finally, a numerical experiment is conducted to confirm our theoretical analysis. Here, h is the spatial subdivision parameter, and τ is the time step.

Mathematics subject classification: 65N15, 65N30.
Key words: KGSEs with damping mechanism, Linearized BDF Galerkin FEM, Optimal error estimates, Superconvergence.

1. Introduction

Consider the following initial-boundary value problem of the KGSEs with damping mechanism [1]:

$$\begin{cases} \mathrm{i}u_t+\Delta u+\mathrm{i}\nu u+u\phi=0, & (X,t)\in\Omega\times(0,T], \\ \phi_{tt}+\gamma\phi_t-\Delta\phi+\phi-|u|^2=0, & (X,t)\in\Omega\times(0,T], \\ u(X,t)=0,\phi(X,t)=0, & (X,t)\in\ \partial\Omega\times(0,T], \\ (u,\phi,\phi_t)(X,0)=(u_0,\phi_0,\phi_1)(X), & X\in\Omega, \end{cases} \tag{1.1}$$

in which $X=(x,y),T<+\infty$ and $\Omega\subset\mathbb{R}^2$ is a convex bounded domain with the boundary $\partial\Omega$ and $\mathrm{i}=\sqrt{-1}$ is the imaginary unit. $u:\Omega\to\mathbb{C}$ is a complex-valued function, which describes a system of conserved complex nucleon fields interacting with ϕ which describes neutral real scalar meson fields. u_0 is a given complex-valued function, ϕ_0 and ϕ_1 are two given real-valued functions. The parameters ν and γ are positive constants that describe the damping mechanism of the original model. Specifically, when $\nu=0$ and $\gamma=0$, the system (1.1) degenerates to the standard KGSEs

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A series of mathematical studies have been devoted to KGSEs, such as the analysis of the global solution, singular limit and asymptotic behavior of the solutions in [1-7]. Along the numerical front, diverse numerical methods have been proposed including conservative spectral or pseudo-spectral methods [8-10, 11, 12], conservative finite difference methods [13, 14, 15], orthogonal spline collocation method [16], FEMs [17] and the symplectic and multi-symplectic methods [18, 19] etc. Especially, a Fourier pseudo-spectral conservative scheme was proposed in [11] for solving the 2-dimensional nonlinear KGSEs and the convergence with order $(N^{-r}+\tau^2)$ in the discrete L^2-norm was proved, where N is the number of nodes and τ is the time-step size. In [12], the structure-preserving numerical schemes for the 2-dimensional space fractional KGSE were investigated with averaged vector field methods. In [15], two linear conservative finite difference schemes were discussed, and optimal H^2 error bounds were derived without any time-step restriction condition in 2-dimensional and 3-dimensional cases. In [17], a conservative FEM was established and superconvergence results of the second-order accuracy both in time and space were obtained for 2-dimensional case.

In contrast, to the best of our knowledge, there are few reports on numerical approaches for the KGSEs with damping mechanism except for [9, 20, 21]. In [9], efficient unconditionally stable and accurate Fourier pseudo-spectral methods were presented for approximations of the KGSEs with damping terms. In addition, a semi-linearized, decoupled time-stepping method was developed in [20] for solving (1.1). And a fully discrete finite difference scheme for model (1.1) in 3-dimensional case was proposed and analyzed in [21]. However, how to solve the problem (1.1) with FEM is an essential problem to be solved.

In this paper, we strive to achieve optimal error estimates and superconvergence results without any time-step restriction condition for equations (1.1) with BDF-FEM because the BDF type schemes are multi-step methods and unconditionally stable [22, 23], and can achieve high order accuracy without increasing the computation significantly [24]. Different from the previous work, a parabolic system is developed by introducing $\psi=\phi_t$ and a BDF time discrete scheme is constructed. There are three keys in our methodology. First, to get rid of the time-step restriction condition of linearized schemes arising from nonlinearity terms, we employ the approach proposed in [25, 26] to split the error into two parts, i.e., the temporal error and the spatial error. Second, to overcome the trouble from the time derivative $D_\tau\sigma^n$, we utilize a novel splitting technique to make the time-step τ transferred for one part of the inner product to another. Through such way, we avoid grid-ratio conditions and derive optimal error estimates in L^2 and H^1-norm. Third, we adopt a so-called "lifting" skill to deal with the absence of $\|\bar{\partial}_\tau\xi^n\|_0$ on the left hand of error equations (see (4.29)–(4.30)) caused by the imaginary unit $\mathrm{i}=\sqrt{-1}$. Besides, the combination idea of the Ritz projection and the finite element interpolation is employed to obtain the unconditional superconvergence results. Based on above achievements, we avoid grid-ratio conditions for the problem (1.1) with BDF-FEM.

The outline of the article is arranged in the following way. In Section 2, a linearized BDF scheme is performed after introducing the finite element and some preliminaries. In Section 3, a corresponding time discrete system was proposed and the temporal error with order $O(\tau^2)$ is deduced. Section 4 contains two parts: optimal error estimates in L^2 and H^1-norms are deduced in Subsection 4.1., and superconvergence results in H^1-norm are derived in Subsection 4.2. Numerical examples are given to confirm our theoretical analysis in Section 5 and a conclusion is presented in Section 6, respectively.

In our paper, any C, with or without superscripts and subscripts, denotes a generic positive constant, not necessarily the same at different occurrences, which is always dependent on the