Journal of Computational Mathematics
Vol.41, No.2, 2023, 246–262.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2110-m2021-0157

ESTIMATION AND UNCERTAINTY QUANTIFICATION FOR PIECEWISE SMOOTH SIGNAL RECOVERY*

Victor Churchill
Department of Mathematics, The Ohio State University, Columbus, OH, USA
Email: churchill.77@osu.edu
Anne Gelb[1)]
Department of Mathematics, Dartmouth College, Hanover, NH, USA
Email: Anne.E.Gelb@dartmouth.edu

Abstract

This paper presents an application of the sparse Bayesian learning (SBL) algorithm to linear inverse problems with a high order total variation (HOTV) sparsity prior. For the problem of sparse signal recovery, SBL often produces more accurate estimates than maximum *a posteriori* estimates, including those that use ℓ_1 regularization. Moreover, rather than a single signal estimate, SBL yields a full posterior density estimate which can be used for uncertainty quantification. However, SBL is only immediately applicable to problems having a *direct* sparsity prior, or to those that can be formed via synthesis. This paper demonstrates how a problem with an HOTV sparsity prior can be formulated via synthesis, and then utilizes SBL. This expands the class of problems available to Bayesian learning to include, e.g., inverse problems dealing with the recovery of piecewise smooth functions or signals from data. Numerical examples are provided to demonstrate how this new technique is effectively employed.

Mathematics subject classification: 62F15, 65C60, 65F22, 94A12.
Key words: High order total variation regularization, Sparse Bayesian learning, Analysis and synthesis, Piecewise smooth function recovery.

1. Introduction

Many real-world phenomena give rise to piecewise smooth signals [29]. As such, their recovery from measurement data is a well-studied inverse problem [36]. Particular attention has been paid to piecewise smooth signal or function recovery from Fourier or spectral data [20–22]. The standard approach to piecewise smooth signal recovery is to minimize a least squares cost function with some form of ℓ_1-norm-based high order total variation (HOTV) regularization [6, 9, 28, 32]. This is well-known to encourage sparsity in the approximate edge domain of the function. While more advanced techniques exist, e.g. total generalized variation [6], in practice this regularization of the approximate edge domain can be achieved in the simplest form by penalizing the gradient domain of the signal using the HOTV operator $\boldsymbol{T}_m \in \mathbb{R}^{(N-m)\times N}$, a finite difference approximation to the mth gradient. Hence we limit our discussion to HOTV, and in particular HOTV orders $m = 1, 2, 3$, and note that our method is easily adapted for $m \geq 4$. Using such an operator is common for inverse problems in image processing when one

* Received June 1, 2021 / Revised version received October 8, 2021 / Accepted October 19, 2021 / Published online November 14, 2022 /
1) Corresponding author

has a prior belief that the signal of interest being recovered is approximately piecewise polynomial of order $m-1$ [1]. This approach has been useful in various applications, for rexample to improve robustness in synthetic aperture radar imaging [33], to recover fine details in electron tomography imaging [34], as well as in interior tomography [42], and MRI [27].

The main contribution of this paper is an alternative Bayesian learning based approach for inverse problems where an HOTV sparsity prior is appropriate. This expands the class of problems available to this strong class of methods which provides a full posterior density estimate rather than a single point estimate. Because the sparsity assumption for piecewise smooth signal reconstruction is typically viewed in the analysis formulation, i.e. $\boldsymbol{T}_m\boldsymbol{x} = \boldsymbol{s}$ with $\boldsymbol{x}$ the signal of interest and $\boldsymbol{s}$ the sparse representation, Bayesian learning is not immediately applicable. To this end, we note that a group of methods has been developed to apply one or more of some types of analysis operators in a Bayesian learning framework, see e.g. [3,10–12,19]. However, the methods in these papers differ from what is considered here in that the analysis operators they examine are limited to square filter matrices. Moreover, the techniques exclusively employ TV regularization, restricting them to first order, although a more sophisticated spatially-variant TV implementation is also considered in some cases. In addition, the forward operators used are also square and limited to blurring applications. This is required to ensure invertibility of the resulting covariance matrix. We address this issue later in Section 4.2. In particular, since the mth order total variation operator $\boldsymbol{T}_m$ is not square and therefore not invertible, and we wish to also consider underdetermined forward operators, another approach is required. In what follows, our approach is to form an approximately equivalent synthesis formulation of the form $\boldsymbol{x} = \boldsymbol{V}\boldsymbol{s}$ in order to effectively reduce the problem to sparse signal recovery. Since sparse Bayesian learning (SBL) [38], is then applicable and has been shown to be more effective than many other methods for sparse signal recovery [23,25], then one can expect better performance in this synthesis construction as well. Our procedure involves a modification from [30] to the analysis operators $\boldsymbol{T}_m$ to make these operators full rank and therefore invertible. This ultimately enables the use of a Bayesian learning algorithm for inverse problems with a HOTV sparsity prior like piecewise smooth signal recovery.

The rest of this paper is organized as follows. Section 2 reviews sparse signal recovery using a maximum a posteriori estimate, and describes how both the synthesis and analysis approaches are typically employed to recover signals that are sparse in a transform domain (e.g. the HOTV domain). Section 3 explains how to formulate a synthesis approach for the HOTV analysis operators via the technique introduced in [30]. Motivated by its success for sparse signal recovery, in Section 4 we demonstrate how SBL specifically can be applied to synthetic HOTV. Numerical examples are implemented in Section 5, where we demonstrate that this approach, which we call high order total variation Bayesian learning (HOTVBL), outperforms the standard ℓ_1 norm based HOTV regularization. Some concluding remarks and ideas for future investigations are provided in Section 6.

2. Background

2.1. Sparse signal recovery

Let $\boldsymbol{x} \in \mathbb{R}^N$ be a sparse signal with $k \ll N$ of its elements nonzero. We seek to recover $\boldsymbol{x}$ from measurements

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{n}, \tag{2.1}$$