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ON THE EXPLICIT TWO-STAGE FOURTH-ORDER ACCURATE TIME DISCRETIZATIONS*

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Abstract

This paper continues to study the explicit two-stage fourth-order accurate time discretizations [5, 7]. By introducing variable weights, we propose a class of more general explicit one-step two-stage time discretizations, which are different from the existing methods, e.g. the Euler methods, Runge-Kutta methods, and multistage multiderivative methods etc. We study the absolute stability, the stability interval, and the intersection between the imaginary axis and the absolute stability region. Our results show that our two-stage time discretizations can be fourth-order accurate conditionally, the absolute stability region of the proposed methods with some special choices of the variable weights can be larger than that of the classical explicit fourth- or fifth-order Runge-Kutta method, and the interval of absolute stability can be almost twice as much as the latter. Several numerical experiments are carried out to demonstrate the performance and accuracy as well as the stability of our proposed methods.

Mathematics subject classification: 65L05, 65L06, 65L20.
Key words: Multistage multiderivative methods, Runge-Kutta methods, Absolute stability region, Interval of absolute stability.

1. Introduction

The explicit two-stage fourth-order accurate time discretizations are studied in [5, 7] and successfully applied to the nonlinear hyperbolic conservation laws. They belong to the two-derivative Runge-Kutta methods, see [1,3,6]. In comparison with the explicit four-stage fourth-order accurate Runge-Kutta method, they only call the time-consuming exact or approximate Riemann solver and the initial reconstruction with the characteristic decomposition twice at each time step, which is half of the former.

For the sake of simplicity, let us consider the initial-value problem of the first-order ordinary differential equation (ODE)

$$u'(t) = L(t, u), \quad t \in [0, T]; \qquad u(0) = u_0, \tag{1.1}$$

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where u is scalar and $L(t,u)$ is linear or nonlinear with respect to u. Assume that the solution u of (1.1) is a sufficiently smooth function of t and L is also smooth, and give a partition of the time interval by $t_{n+1} = t_n + \tau$, $n \in \mathbb{Z}^+ \cup \{0\}$, where τ denotes the time step-size. The Taylor series expansion of u in t reads

$$\begin{aligned} u^{n+1} &= \left(u + \tau \frac{du}{dt} + \frac{\tau^2}{2!}\frac{d^2u}{dt^2} + \frac{\tau^3}{3!}\frac{d^3u}{dt^3} + \frac{\tau^4}{4!}\frac{d^4u}{dt^4}\right)^n + \mathcal{O}(\tau^5) \\ &= \left(u + \tau L(t,u) + \frac{\alpha\tau^2}{2}\mathcal{D}_t L(t,u)\right)^n \\ &\quad + \frac{(1-\alpha)\tau^2}{2}\left(\frac{d^2}{dt^2}\left(u + \frac{\tau}{3(1-\alpha)}L(t,u) + \frac{\tau^2}{12(1-\alpha)}\mathcal{D}_t L(t,u)\right)\right)^n + \mathcal{O}(\tau^5), \end{aligned} \tag{1.2}$$

where $\mathcal{D}_t = \partial_t + L\partial_u$ and α does not depend on t, u.

Based on the additive decomposition (1.2) with $\alpha = 1/3$, the explicit two-stage fourth-order time-accurate discretization [5] can be implemented as follows

$$\begin{aligned} u^* &= u^n + \frac{\tau}{2}L(t^n,u^n) + \frac{\tau^2}{8}(\mathcal{D}_t L)(t^n,u^n), \\ u^{n+1} &= u^n + \tau L(t^n,u^n) + \frac{\tau^2}{6}\Big[(\mathcal{D}_t L)(t^n,u^n) + 2(\mathcal{D}_t L)(t^n + \tau/2, u^*)\Big], \end{aligned} \tag{1.3}$$

which can also be found in [3, Section 3], [1, Section 3.2] and [6, Section 1]. For a general choice of α that $\alpha = \alpha(\hat{\tau})$ is a differentiable function of $\hat{\tau} = \tau^p$, $p \geq 1$, and satisfies $\alpha = 1/3 + \mathcal{O}(\hat{\tau})$ and $\alpha \neq 1$, the general two-stage fourth-order time-accurate discretization [7] can be given as follows

$$\begin{aligned} u^* &= u^n + \frac{\tau}{3(1-\alpha)}L(t^n,u^n) + \frac{\tau^2}{12(1-\alpha)}(\mathcal{D}_t L)(t^n,u^n), \\ u^{n+1} &= u^n + \tau L(t^n,u^n) + \frac{\tau^2}{2}\left[\alpha(\mathcal{D}_t L)(t^n,u^n) + (1-\alpha)(\mathcal{D}_t L)\left(t^n + \frac{\tau}{3(1-\alpha)}, u^*\right)\right], \end{aligned} \tag{1.4}$$

which are not mentioned in the literature. If applying methods (1.3) and (1.4) to the model problem $u'(t) = \lambda u(t)$ and taking θ and z as u^{n+1}/u^n and $\tau\lambda$, respectively, then we can directly obtain their stability polynomial

$$\pi(\theta, z) = \theta - \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4\right),$$

with a single root $\theta_1 = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$, which is exactly the same as that of the (classical) explicit four-stage fourth-order accurate Runge-Kutta method. For the absolute stability [2, 4], one requires

$$|\theta_1| \leq 1, \quad \text{i.e.} \quad \left|1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4\right| \leq 1.$$

It is worth noting that there exist some examples of inequivalent definitions of the region of absolute stability of a numerical method for ODEs in the literature[1)] .

Does there exist any explicit two-stage fourth-order accurate time discretization with a larger region of absolute stability? The aim of this paper is to answer this question and to propose

[1)] http://vmm.math.uci.edu/ODEandCM/StabiltyRegionDefinitions/StabilityRegionDefinitions.html