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DISCRETE ENERGY ANALYSIS OF THE THIRD-ORDER VARIABLE-STEP BDF TIME-STEPPING FOR DIFFUSION EQUATIONS*

Hong-lin Liao[1)]
Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China;
Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 211106, China
Email: liaohl@nuaa.edu.cn, liaohl@csrc.ac.cn

Tao Tang
Division of Science and Technology, BNU-HKBU United International College, Zhuhai 519087, China
Email: tangt@sustech.edu.cn

Tao Zhou
Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
Email: tzhou@lsec.cc.ac.cn

Abstract

This is one of our series works on discrete energy analysis of the variable-step BDF schemes. In this part, we present stability and convergence analysis of the third-order BDF (BDF3) schemes with variable steps for linear diffusion equations, see, e.g., [SIAM J. Numer. Anal., 58:2294-2314] and [Math. Comp., 90: 1207-1226] for our previous works on the BDF2 scheme. To this aim, we first build up a discrete gradient structure of the variable-step BDF3 formula under the condition that the adjacent step ratios are less than 1.4877, by which we can establish a discrete energy dissipation law. Mesh-robust stability and convergence analysis in the L^2 norm are then obtained. Here the mesh robustness means that the solution errors are well controlled by the maximum time-step size but independent of the adjacent time-step ratios. We also present numerical tests to support our theoretical results.

Mathematics subject classification: 65M06, 65M12.
Key words: Diffusion equations, Variable-step third-order BDF scheme, Discrete gradient structure, Discrete orthogonal convolution kernels, Stability and convergence.

1. Introduction

In this paper, we aim to develop a discrete energy technique for the stability and convergence of three-step backward differentiation formula (BDF3) with variable time-steps. To this end, we consider the linear reaction-diffusion problem in a bounded convex domain Ω,

$$\partial_t u - \varepsilon \Delta u = \kappa(x) u + f(t, x) \quad \text{for } x \in \Omega \text{ and } 0 < t < T, \tag{1.1}$$

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1) Corresponding author

subject to the Dirichlet boundary condition $u = 0$ on a smooth boundary $\partial\Omega$, and the initial data $u(0,x) = u_0$ for $x \in \Omega$. We assume that the diffusive coefficient $\varepsilon > 0$ is a constant and the reaction coefficient $\kappa(x)$ is smooth and bounded by $\kappa^* > 0$.

The BDF schemes are widely used for stiff or differential-algebraic problems [7,8]. Recently, they were also applied for simulating hyperbolic systems with multiscale relaxation [1] and stiff kinetic equations [5]. For such applications, BDF schemes with variable steps are shown to be computationally efficient in capturing the multi-scale time behaviors [1,2,4,6,8,12,19]. However, rigorous theoretically analysis (stability and convergence) for variable-step BDF schemes is challenging. This motivates our serious works on this topic, and one can find our previous works on the variable-step BDF2 scheme [11, 13, 14, 16].

To begin, we consider the temporal mesh

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

with the variable time-step

$$\tau_k := t_k - t_{k-1}, \quad 1 \le k \le N.$$

The maximum step size and the adjacent time-step ratios are defined respectively as

$$\tau := \max_{1\le k\le N} \tau_k, \quad r_k := \frac{\tau_k}{\tau_{k-1}} \ \text{ for } \ 2 \le k \le N.$$

For any sequences $\{v^n\}_{n=0}^N$, we denote $\partial_\tau v^n := (v^n - v^{n-1})/\tau_n$. Let $\Pi_{n,3}v$ be the Newton interpolating polynomial of a function v over the nodes t_{n-3}, t_{n-2}, t_{n-1} and t_n. By taking $v^n = v(t_n)$, the variable-step BDF3 formula [3, 10] is defined by $D_3v^n := (\Pi_{n,3}v)'(t_n)$ for $n \ge 3$, which yields

$$D_3v^n = d_0(r_n, r_{n-1})\partial_\tau v^n + d_1(r_n, r_{n-1})\partial_\tau v^{n-1} + d_2(r_n, r_{n-1})\partial_\tau v^{n-2}, \tag{1.2}$$

where the variable ceofficients d_0, d_1 and d_2 are defined by

$$d_0(x,y) := \frac{1+2x}{1+x} + \frac{xy}{1+y+xy}, \tag{1.3}$$

$$d_1(x,y) := -\frac{x}{1+x} - \frac{xy}{1+y+xy} - \frac{xy^2}{1+y+xy}\frac{1+x}{1+y}, \tag{1.4}$$

$$d_2(x,y) := \frac{xy^2}{1+y+xy}\frac{1+x}{1+y}, \qquad \text{for } x, y \ge 0. \tag{1.5}$$

Without losing the generality, we assume that the discrete solution u^1 and u^2 are given. We now consider a time-discrete solution for the diffusion equations, $u^k(x) \approx u(t_k, x)$ for $x \in \Omega$, by the following variable-step BDF3 time-stepping scheme

$$D_3u^k = \varepsilon\Delta u^k + \kappa u^k + f^k \quad \text{for } 3 \le k \le N, \tag{1.6}$$

where $f^k(x) = f(t_k, x)$.

To the best of our knowledge, there are very few theoretical results on the variable-step BDF3 scheme in literature. For linear diffusion problems, Calvo and Grigorieff [3] established the following L^2 norm stability estimate under the step-ratio condition $r_k < 1.199$,

$$\|u^n\| \le C\exp(C\Gamma_n)\Big(\|u_0\| + \sum_{j=1}^n \tau_j \|f^j\|\Big) \quad \text{for } n \ge 1,$$