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STOCHASTIC VARIATIONAL INEQUALITY APPROACHES TO THE STOCHASTIC GENERALIZED NASH EQUILIBRIUM WITH SHARED CONSTRAINTS*

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Abstract

In this paper, we consider the generalized Nash equilibrium with shared constraints in the stochastic environment, and we call it the stochastic generalized Nash equilibrium. The stochastic variational inequalities are employed to solve this kind of problems, and the expected residual minimization model and the conditional value-at-risk formulations defined by the residual function for the stochastic variational inequalities are discussed. We show the risk for different kinds of solutions for the stochastic generalized Nash equilibrium by the conditional value-at-risk formulations. The properties of the stochastic quadratic generalized Nash equilibrium are shown. The smoothing approximations for the expected residual minimization formulation and the conditional value-at-risk formulation are employed. Moreover, we establish the gradient consistency for the measurable smoothing functions and the integrable functions under some suitable conditions, and we also analyze the properties of the formulations. Numerical results for the applications arising from the electricity market model illustrate that the solutions for the stochastic generalized Nash equilibrium given by the ERM model have good properties, such as robustness, low risk and so on.

Mathematics subject classification: 90C15, 90C33.
Key words: Stochastic generalized Nash equilibrium, Stochastic variational inequalities, Expected residual minimization model, CVaR formulation, Gradient consistency.

1. Introduction

Nash equilibrium (NE) which has a wide range of applications was introduced in [25]. The standard normal form of a noncooperative game usually assumes that the feasible set of each player is independent of the other players. When the feasible set or the feasible strategy set depends on the others' strategy, the equilibrium is called the generalized Nash equilibrium (GNE) [1, 16, 23, 32]. GNE naturally has lots of applications arising from the complex and important economical and engineering systems.

We suppose that GNE consists of N players, and the variable $x^\nu \in R^{n_\nu}$ denotes the ν-th player's strategy. We let x be the vector which is formed by all these decision variables:

$$x := \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^N \end{pmatrix} \in R^n, \qquad n = n_1 + \cdots + n_N,$$

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and $x^{-\nu}$ denotes the vector formed by all the players' strategies except those of player ν. Sometimes, we use $(x^\nu, x^{-\nu})$ to instead of x. The decision variable of player ν must belong to a feasible set $X_\nu(x^{-\nu}) \subseteq R^{n_\nu}$, which depends on the other players' strategies. The problem is described as follows:

$$\begin{aligned} \min_{x^\nu} \quad & \theta_\nu(x^\nu, x^{-\nu}) \\ \text{s.t.} \quad & x^\nu \in X_\nu(x^{-\nu}). \end{aligned} \tag{1.1}$$

For any $x^{-\nu}$, we denote $S_\nu(x^{-\nu})$ as the solution set of the problem (1.1), and the GNE is to find a vector $\bar{x}$ such that

$$\bar{x}^\nu \in S_\nu(\bar{x}^{-\nu}) \quad \text{for all } \nu.$$

For the GNE, such point $\bar{x}$ is called a Nash equilibrium or a solution of the GNE.

Suppose that for the player ν, θ_ν is continuously differentiable in x, and for each player ν, the function $\theta_\nu(\cdot, x^{-\nu})$ is convex in x^ν. Then, the quasi-variational inequality which can represent the GNE is to find a vector $x^\nu \in X_\nu(x^{-\nu})$ such that

$$(y^\nu - x^\nu)^T F^\nu(x^\nu) \geq 0 \quad \text{for all } y^\nu \in X_\nu(x^{-\nu}), \tag{1.2}$$

where $F^\nu(x^\nu) := \nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu})$.

The GNE with shared constraints is an important case of this kind of problem [7, 12, 32], and the feasible set of which is defined by

$$X_\nu(x^{-\nu}) := \{x^\nu | (x^\nu, x^{-\nu}) \in X\} \quad \text{and} \quad X := \prod_{\nu=1}^{\mathrm{N}} X_\nu(\mathrm{x}^{-\nu}).$$

The GNE with shared constraints can be solved by finding a solution of a single variational inequality (VI) [12, 13], which is given as follows:

$$(y - x)^T F(x) \geq 0 \quad \text{for all } y \in X, \tag{1.3}$$

where

$$F(x) := \begin{pmatrix} \nabla_{x^1}\theta_1(x) \\ \nabla_{x^2}\theta_2(x) \\ \vdots \\ \nabla_{x^N}\theta_N(x) \end{pmatrix}.$$

When θ_ν is continuously differentiable in x, and for each player ν, the function $\theta_\nu(\cdot, x^{-\nu})$ is pseudo-convex in x^ν, from Theorem 2.1 in [12] we know that any solution of the corresponding VI (1.3) is a solution of the GNE (1.1). But in general, solutions of the GNE may not be those of VI.

The GNE mentioned by the above papers and formulations is deterministic. However, in many real applications in engineering, management and science, the players have to make the decisions in the stochastic environment, and we call it stochastic generalized Nash equilibrium (SGNE). A general way to study the SGNE is to use the expected value of the payoffs to define the SGNE formulation [17, 18, 21, 27, 36], and the sample average approximation method of