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INVERSE CONDUCTIVITY PROBLEM WITH INTERNAL DATA*

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Abstract

This paper concerns the reconstruction of a scalar coefficient of a second-order elliptic equation in divergence form posed on a bounded domain from internal data. This problem finds applications in multi-wave imaging, greedy methods to approximate parameter-dependent elliptic problems, and image treatment with partial differential equations. We first show that the inverse problem for smooth coefficients can be rewritten as a linear transport equation. Assuming that the coefficient is known near the boundary, we study the well-posedness of associated transport equation as well as its numerical resolution using discontinuous Galerkin method. We propose a regularized transport equation that allow us to derive rigorous convergence rates of the numerical method in terms of the order of the polynomial approximation as well as the regularization parameter. We finally provide numerical examples for the inversion assuming a lower regularity of the coefficient, and using synthetic data.

Mathematics subject classification: 35R30, 65N21.
Key words: Inverse problems, Multi-wave imaging, Static transport equation, Internal data, Diffusion coefficient, Stability estimates, Regularization.

1. Introduction

Let Ω be a C^6-smooth bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with boundary Γ. Let $\nu(x)$ be the outward normal vector at $x \in \Gamma$, and $d = \sup_{x,y\in\Omega} \|x - y\|$ be the diameter of Ω. We set, for $\eta \in (0, d)$, $\sigma_0 \in W^{2,\infty}(\Omega)$, $\Omega_\eta = \{x \in \Omega, \mathrm{dist}(x, \Gamma) > \eta\}$, and $0 < k_1 < k_2$,

$$\Sigma = \{\sigma \in W^{2,\infty}(\Omega);\ \sigma = \sigma_0 \ \text{ in } \Omega \setminus \overline{\Omega_\eta},\ k_1 \le \sigma,\ \|\sigma\|_{W^{2,\infty}(\Omega)} \le k_2\}.$$

Let g be fixed in $H^{\frac{7}{2}}(\Gamma)$, and satisfy $\int_\Gamma g dx = 0$. Then, according to the classical elliptic regularity theory

$$\mathrm{div}(\sigma\nabla u) = 0 \text{ in } \Omega, \qquad \sigma\partial_\nu u = g \text{ on } \Gamma, \ \int_\Omega u_\sigma dx = 0, \tag{1.1}$$

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has a unique solution $u_\sigma \in H^5(\Omega)$ [1], and there exists a constant $c = c(\Sigma, \Omega) > 0$ such that

$$\|u_\sigma\|_{H^5(\Omega)} \leq c. \tag{1.2}$$

The goal of this work is to study the following inverse problem (IP): Given σ_0 and the interior data $u_\sigma|_\Omega$, to reconstruct the conductivity $\sigma|_\Omega$.

This inverse problem is of importance in many different scientific and engineering fields including photoacoustic tomography, studies of effective properties of composite materials, and approximation of parametric partial differential equations. Photoacoustic tomography is a recent hybrid imaging modality that couples diffusive optical waves with ultrasound waves to achieve high-resolution imaging of optical properties of biological tissues [2–7]. The inverse problem (IP) appears in the second inversion, called quantitative photoacoustic tomography, where the derived internal data is used to recover the optical coefficients of the sample [8, 9]. Motivated by the search for sharp bounds on the effective moduli of composites many researchers have considered the problem of characterizing mathematically among all the gradient fields those solving the equation (1.1) for some function σ within the set Σ. In the context of approximation of parameter-dependent elliptic problems by greedy algorithms the inverse problem (IP) has been considered with infinitely many interior data available [10]. Hence solving the inverse problem with a single datum may reduce the dimensionality of the set of parameters used to accurately approximate a targeted compact set of solutions [11].

Given σ_0 and the interior data $u_\sigma|_\Omega$, the inverse problem can be recasted as a linear steady transport equation satisfied by $\sigma \in \Sigma$,

$$\nabla\sigma \cdot \nabla u_\sigma + (\Delta u_\sigma)\sigma = 0 \ \text{ in } \ \Omega.$$

The steady transport equation is one of the basic equations in mathematical physics. It is widely used in fluid mechanics, for example to model mass transfer [12]. From the mathematical point of view there are several results addressing the well-posdeness of the equation. In order to briefly review some of these results we introduce suitable boundary conditions. To do so we split the boundary of Γ into three disjoint parts, the inflow Γ_{in}, the outflow set Γ_{out}, and the characteristic set Γ_0, defined by

$$\Gamma_{\text{in}} = \{x \in \Gamma : \nabla u_\sigma \cdot \nu < 0\}, \quad \Gamma_{\text{out}} = \{x \in \Gamma : \nabla u_\sigma \cdot \nu > 0\}, \quad \Gamma_0 = \Gamma \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}}). \tag{1.3}$$

Assuming that ∇u_σ never vanishes in Ω and using the method of characteristics, one can easily show that the system

$$\nabla\sigma \cdot \nabla u_\sigma + (\Delta u_\sigma)\sigma = 0 \quad \text{in } \Omega, \qquad \sigma = \sigma_0 \quad \text{on } \Gamma_{\text{in}}, \tag{1.4}$$

admits a unique solution. The method of characteristics can not be applied when the set of characteristic curves has a complex structure, for example when ∇u_σ vanishes. In order to overcome this difficulty, many works have considered the case where the lower order term dominates the transport term. In this framework the theory of linear steady transport equations becomes part of a more general theory of degenerate elliptic equations ([13–15], see also Chapter 12 in [12] and references therein). Let $\kappa > 0$ be a fixed constant. When $n = 3$, and assuming that the interior data u_σ verifies

$$\inf_{x\in\Omega} |\Delta u_\sigma(x)| > \kappa > 0, \tag{1.5}$$