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THEORETICAL ANALYSIS OF THE REPRODUCING KERNEL GRADIENT SMOOTHING INTEGRATION TECHNIQUE IN GALERKIN MESHLESS METHODS*

Xiaolin Li
School of Mathematical Sciences, Chongqing Normal University, Chongqing 400047, China
Email: lxlmath@163.com

Abstract

Numerical integration poses greater challenges in Galerkin meshless methods than finite element methods owing to the non-polynomial feature of meshless shape functions. The reproducing kernel gradient smoothing integration (RKGSI) is one of the optimal numerical integration techniques in Galerkin meshless methods with minimum integration points. In this paper, properties, quadrature rules and the effect of the RKGSI on meshless methods are analyzed. The existence, uniqueness and error estimates of the solution of Galerkin meshless methods under numerical integration with the RKGSI are established. A procedure on how to choose quadrature rules to recover the optimal convergence rate is presented.

Mathematics subject classification: 65D32, 65N15, 65N30.
Key words: Galerkin meshless method, Numerical integration, Quadrature rule, Error estimates, Element-free Galerkin method, Degree of precision.

1. Introduction

Galerkin meshless (or meshfree) methods have developed rapidly in the past three decades [1–3]. Compared with finite element methods (FEMs), shape functions in meshless methods are constructed with high smoothness and computational accuracy by using scattered nodes instead of elements [2–5]. Nevertheless, since meshless shape functions are usually not polynomials, enough integration points are necessitated in the often used Gauss quadrature rules to get acceptable accuracy [6–8]. High order Gauss quadrature rules not only greatly increase the computational cost, but also cannot recover optimal convergence [7–10]. Thus, the efficient computation of integrals in Galerkin meshless methods poses considerable difficulty and is crucial to ensure the accuracy and convergence.

The effect of numerical integration on Galerkin meshless methods has been studied numerically, and some numerical integration techniques have been proposed to circumvent the integration difficulties [8–14]. Although some works [3, 15–19] have been done for the theoretical error analysis of Galerkin meshless methods, these estimates are solely bounded by the meshless approximation error. Up to date, only a few papers deal with the theoretical analysis of numerical integration techniques in meshless methods. Under a "zero row sum" assumption, Babuška *et. al.* pioneered the theoretical analysis in this area for linear basis functions [7]. This result is generalized in Refs. [13, 14] for arbitrary order basis functions, where the quadrature rules are assumed to satisfy a "discrete Green's identity" (also called an "integration constraint"). In Refs. [13, 14], quadrature rules are not explicitly constructed and a system of algebraic

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equations needs to be solved with additional effort in each influence domain to attain integration points and weights. These quadrature rules violate the symmetry of variational formulations and lead to non-symmetric stiffness matrices. Theoretical analysis shows that the accuracy of these quadrature rules will affect the convergence rate and then the associated meshless methods may not recover optimal convergence.

The reproducing kernel gradient smoothing integration (RKGSI) [9] is an efficient numerical integration technique for Galerkin meshless methods with arbitrary order basis functions. In the RKGSI, smoothed gradients of meshless shape functions are introduced in a reproducing kernel form to satisfy the integration constraint of variational formulations, and explicit quadrature rules with the minimum number of integration points are constructed by forming a linear integer programming problem and solving it with the branch and bound algorithm. The smoothed gradients are used to discretize variational formulations and thus, the time-consuming computation of meshless gradients is completely avoided. Unlike the quadrature rules in Refs. [11,13,14], all integration points and weights in the RKGSI can be explicitly constructed in the reference space. Besides, the RKGSI leads to symmetric stiffness matrices.

Numerical results [9, 10] indicate that the RKGSI can obtain optimal convergence and outperforms Gauss quadrature rules and other integration techniques [11, 12] in terms of accuracy, convergence and efficiency. The RKGSI is one of the optimal numerical integration techniques in Galerkin meshless methods with minimum integration points. However, theoretical results of the RKGSI have not been found in the literature. Recently, Wu and Wang [10] presented a theoretical accuracy analysis of Galerkin meshless methods accounting for numerical integration, but the properties of the smoothed gradients and the effect of both the RKGSI and the smoothed gradients on Galerkin meshless methods have not been analyzed theoretically.

This paper is devoted to the theoretical analysis of the RKGSI in Galerkin meshless methods. Firstly, properties and error of the smoothed gradients are discussed theoretically. Secondly, quadrature rules in the RKGSI are redesigned by establishing fundamental quadrature criteria with necessary degrees of precision that the quadrature has to satisfy. Then, as in FEMs [20,21], quadrature rules can be routinely constructed in the reference space to accurately integrate complete polynomials up to the least degrees of precision. Compared with Ref. [9], the procedure of determining integration points and weights in this paper is more simple and convenient to use. Thirdly, the existence and uniqueness of the solution of the Galerkin meshless method with smoothed gradients and numerical integration are established, which ensure the validity of the quadrature rules in the RKGSI. Although smoothed gradients have been used in many Galerkin meshless methods [8–12], the relevant theoretical analysis is still in the initial stage. Fourthly, taking the element-free Galerkin method [1, 3, 11, 22, 23] as an example, a framework for error estimates of Galerkin meshless methods under numerical integration with the RKGSI is set up. The theoretical error estimation indicates how to choose quadrature rules to recover the optimal convergence rate.

2. Galerkin Meshless Formulation

For the sake of clarity, the Galerkin meshless method considered in this research is specified to the element-free Galerkin method [1], which is one of the most notable and widely used meshless methods using the moving least squares (MLS) approximation [24] to form shape functions. Consider the following model problem in a nonempty and open bounded plane domain Ω with polygonal boundary Γ,