Journal of Computational Mathematics
Vol.41, No.4, 2023, 569–587.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2107-m2021-0114

A LOW ORDER NONCONFORMING MIXED FINITE ELEMENT METHOD FOR NON-STATIONARY INCOMPRESSIBLE MAGNETOHYDRODYNAMICS SYSTEM*

Zhiyun Yu
College of Science, Zhongyuan University of Technology, Zhengzhou 450007, China
Email: 5772@zut.edu.cn
Dongyang Shi[1)]
School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China
Email: shi_dy@zzu.edu.cn
Huiqing Zhu
School of Mathematics and Natural Sciences, University of Southern Mississippi, Hattiesburg, MS 39406, U.S.A
Email: huiqing.zhu@usm.edu

Abstract

A low order nonconforming mixed finite element method (FEM) is established for the fully coupled non-stationary incompressible magnetohydrodynamics (MHD) problem in a bounded domain in 3D. The lowest order finite elements on tetrahedra or hexahedra are chosen to approximate the pressure, the velocity field and the magnetic field, in which the hydrodynamic unknowns are approximated by inf-sup stable finite element pairs and the magnetic field by $H^1(\Omega)$-conforming finite elements, respectively. The existence and uniqueness of the approximate solutions are shown. Optimal order error estimates of $L^2(H^1)$-norm for the velocity field, $L^2(L^2)$-norm for the pressure and the broken $L^2(H^1)$-norm for the magnetic field are derived.

Mathematics subject classification: 65N15, 65N30, 65M60, 65M12.
Key words: Non-stationary incompressible MHD problem, Nonconforming mixed FEM, Optimal order error estimates.

1. Introduction

The incompressible MHD equation which couples Navier-Stokes equations with Maxwell's equations is usually used to describe the flow of a viscous, incompressible, and electrically conducting fluid. The problem has a number of applications such as liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD power generation and MHD ion propulsion [1].

Many studies have been devoted to the numerical analysis of stationary incompressible MHD problems both in 2D and 3D. Compared with the finite difference methods [2, 3], most studies are performed by FEMs [4–22]. The work started with [4], where the inf-sup stable mixed finite elements were used to discretize the velocity field and the pressure, and H^1-conforming finite elements for the magnetic field, respectively. The existence and uniqueness of the solutions of a weak form and a discrete form with inhomogeneous boundary condition were proved, and the convergence analysis was presented, provided that $\Omega \subset R^3$ is either a convex polyhedron or has

* Received April 21, 2021 / Revised version received June 20, 2021 / Accepted July 16, 2021 / Published online November 1, 2022 /
1) Corresponding author

a boundary which is $C^{1,1}$(see, Theorem 6.4. in [4]). For convex polyhedral domains, or domains with a boundary $C^{1,1}$, the convergence analysis of Garlerkin mixed FEMs, stabilized FEMs, an optimal control method, the two-level FEM, the nonlinear Galerkin FEM and the Petrov-Galerkin FEM were investigated in [5, 6, 13], [7, 8], [9], [10], [11] and [12], respectively. Some other numerical schemes in general Lipschitz polyhedral domains were also realized in [14–23], where mixed discrete formulations were proposed to approximate the magnetic field based on $H(curl)$-conforming (edge) elements.

For non-stationary incompressible MHD problems, strong solutions are only known to exist for small times and sufficiently regular data, while weak solutions exist globally. So it is important to develop different numerical schemes for the unsteady situation. The known results including modeling, analysis and numerics are summarized in [1]. Moreover, the long-term dissipativity and unconditional nonlinear stability of time integration algorithms were firstly examined in [24]. From then on, the convergence of iterations of different coupling and decoupling fully discrete schemes towards weak solutions was verified in [25, 26], but they paid no attention to the error estimates of the associated unknown variables. In order to make up for this deficiency, the convergence analysis of locally divergence-free discontinuous Galerkin methods with a second order Runge-Kutta time discretization was presented in [27], and the error estimate was obtained for the the magnetic field in L^2-norm of order $O(\Delta t^2 + h^{m+\frac{1}{2}})$, where m is the degree of the local complete polynomials contained in the approximating space. The stability and error analysis of the semi-discrete and Crank-Nicolson discretization were derived in [28] for the quasi-static MHD equation at the small magnetic Reynolds number R_m. The numerical analysis of the Backward-Euler discretization was studied in [29] for the same equation as that of [28]. But the error estimate of the pressure for this uncoupled unsteady MHD equations is not investigated in [28, 29]. Later on, more and more researchers paid more attention to the fully coupled unsteady MHD equations, see, e.g. [30–37], which deduced optimal error estimates for the all variables by the different discrete schemes. However, all of the analysis mentioned above mainly concentrated on the conforming FEMs.

As we know, nonconforming finite elements, to some extend, are easier to be constructed to satisfy the discrete inf-sup condition compared with conforming finite elements, and the associated methods can be seen somehow between conforming FEMs and discontinuous Galerkin methods. The continuity requirement of conforming FEMs is weakened in nonconforming FEMs but not removed completely from the approximation spaces as it is done in discontinuous Galerkin methods. Furthermore, the use of nonconforming FEMs can avoid the implementation of jumps terms which are essential for discontinuous Galerkin methods [62]. So nonconforming FEMs have been used effectively in flow problems and Maxwell's equations in 2D or 3D such as the convection-dominated transport problem [41, 42], the Stokes equations [43–52], the Navier-Stokes equations [53–60], the conduction-convection problem [61], the Maxwell's equations [63–69] and so on. For the fully coupled stationary MHD equations, nonconforming mixed FEMs were analyzed in [17,70,71]. More precisely, the exactly divergence-free velocity approximations and quasi-optimal order error estimates were derived in [17]. A family of low order nonconforming mixed FEMs were proposed and optimal order error estimates for all variables were obtained in [70, 71],.

As a continuous work, we are interested in the analysis of the nonconforming FEMs for the fully coupled non-stationary MHD equations. The framework of nonconforming mixed FEMs is developed and analyzed in this paper. Firstly, using the new inequality of the trilinear form in the mixed variational formulation for MHD equations, we prove the existence and uniqueness