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TWO-PHASE IMAGE SEGMENTATION BY NONCONVEX NONSMOOTH MODELS WITH CONVERGENT ALTERNATING MINIMIZATION ALGORITHMS *

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Abstract

Two-phase image segmentation is a fundamental task to partition an image into foreground and background. In this paper, two types of nonconvex and nonsmooth regularization models are proposed for basic two-phase segmentation. They extend the convex regularization on the characteristic function on the image domain to the nonconvex case, which are able to better obtain piecewise constant regions with neat boundaries. By analyzing the proposed non-Lipschitz model, we combine the proximal alternating minimization framework with support shrinkage and linearization strategies to design our algorithm. This leads to two alternating strongly convex subproblems which can be easily solved. Similarly, we present an algorithm without support shrinkage operation for the nonconvex Lipschitz case. Using the Kurdyka-Łojasiewicz property of the objective function, we prove that the limit point of the generated sequence is a critical point of the original nonconvex nonsmooth problem. Numerical experiments and comparisons illustrate the effectiveness of our method in two-phase image segmentation.

Mathematics subject classification: 94A08, 90C25, 68U10, 47A52, 65K10.
Key words: Nonconvex nonsmooth regularization, Characteristic function, Box constraints, Support shrinking, Alternating minimization, Kurdyka-Łojasiewicz property, Image segmentation.

1. Introduction

Image segmentation aims to partition an image into some disjoint but meaningful regions. A good segmentation result can be used for many application fields such as object detection, recognition, measurement and tracking. In the past decades, various methods have been proposed to solve this problem [6,8,13,14,29–32,47,49,50,55]. Since two-phase image segmentation is a fundamental and widely studied task, we in this paper focus on proposing nonconvex region-based regularization models with globally convergent algorithms for segmenting an image into a foreground and a background.

Among variational region-based segmentation methods, the most well-known model is perhaps the following Mumford-Shah (MS) model [32]

$$\min_{g,\Gamma} \frac{\beta_1}{2} \int_{\Omega} (g(x) - f(x))^2 \, dx + \beta_2 \int_{\Omega\setminus\Gamma} |\nabla g(x)|^2 dx + |\Gamma|, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded open connected set, Γ is a compact curve in Ω, $f : \Omega \to \mathbb{R}$ is a given image, β_1, β_2 are positive parameters, $g : \Omega \to \mathbb{R}$ is the piecewise smooth approximation of f. This model pursues a piecewise constant or smooth approximation g of f. Due to the non-convexity of (1.1), it is a great challenge to find its minimizers.

A simplified variant of MS is piecewise-constant Mumford-Shah model (PCMS), which is a fundamental case of (1.1). It assumes that f is segmented into some intensity-constant regions [3, 7, 14, 39]. When to segment the Ω into a foreground Σ and a background $\Omega \setminus \Sigma$, the PCMS model becomes

$$\min_{c_1,c_2,\Sigma} \frac{\beta}{2}\left(\int_{\Sigma}(c_1 - f(x))^2 dx + \int_{\Omega\setminus\Sigma}(c_2 - f(x))^2 dx\right) + |\partial\Sigma|, \tag{1.2}$$

where $c_1, c_2 \in \mathbb{R}$ are the average values of pixels inside and outside $\partial\Sigma$, and β is a positive parameter. This model is just the CV model [14] without the area term of Σ. Earlier methods to solve (1.2) include level set techniques [14, 27, 35]. Later, Chan et al. [7, 13] proposed the following relaxation model

$$\min_{c_1,c_2,u(x)\in[0,1]} \frac{\beta}{2}\left(\int_{\Omega} u(x)(c_1 - f(x))^2 dx + \int_{\Omega}(1 - u(x))(c_2 - f(x))^2 dx\right) + \int_{\Omega}|\nabla u(x)|dx, \tag{1.3}$$

where u is a characteristic function. Its value will approach one in the foreground and zero in the background. The second term is total variation of u which measures the length of the boundary. With known c_1, c_2, (1.3) is called a labeling problem that is completely convex, and thresholding the solution of (1.3) with any given value in $[0, 1]$ produces a globally optimal solution to (1.2); see [13]. If c_1, c_2 are unknown, (1.3) is still a nonconvex problem, which is difficult [7, 10, 36] to handle.

By introducing edge indicator weights or high-order regularization, some modifications in [6, 21, 31, 50, 55] were proposed. In [6, 21], the authors combined total variation with edge indicator functions to improve the ability of finding edges. In [55], Euler's elastica regularization is employed to interpolate the missing boundaries automatically without specifying regions. In [50], the authors used the weighted first and second-order regularizers to overcome staircase results with discontinuities and rough boundaries. In [31], a weighted wavelet frame based ℓ_1 regularization was proposed for the low contrast ultrasound image and video segmentation. It alternately updated the characteristic function u and region constants c_1, c_2 and gave convergence analyses of the algorithm based on Kurdyka-Łojasiewicz property.

Recently, many studies [24, 26, 34, 45, 52] indicated that nonconvex regularizations composed of first-order information of images can yield better edge and contrast preservation in image restoration problems. A mathematical explanation by establishing uniform lower bounds for nonzero gradients of recovered images was provided for different nonconvex minimization models [17, 23, 33, 52, 53] in image restoration problems. Motived by these studies, [30] utilized nonconvex TV_p for labeling and segmentation with bias correction in additive image models. They solved the nonconvex u-minimization by the alternating direction method of multipliers (ADMM), and gave a weak convergence result only for the labeling problem. Under the two-stage segmentation framework [8, 9, 49], nonconvex regularization models [11, 15, 22, 25, 48] were proposed to produce an approximation image followed by a thresholding strategy for segmentation. We note that, for the basic model (1.3), there is so far no nonconvex variants in the literature.

To solve minimization problems with multiple blocks of variables, the alternating minimization algorithm [1, 2, 5, 7, 31, 43] is an important technique. For example, in (1.3), we can firstly