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SOLVING NONLINEAR DELAY-DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH SINGULAR PERTURBATION VIA BLOCK BOUNDARY VALUE METHODS*

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Abstract

Block boundary value methods (BBVMs) are extended in this paper to obtain the numerical solutions of nonlinear delay-differential-algebraic equations with singular perturbation (DDAESP). It is proved that the extended BBVMs in some suitable conditions are globally stable and can obtain a unique exact solution of the DDAESP. Besides, whenever the classic Lipschitz conditions are satisfied, the extended BBVMs are preconsistent and pth order consistent. Moreover, through some numerical examples, the correctness of the theoretical results and computational validity of the extended BBVMs is further confirmed.

Mathematics subject classification: 34K50, 60H35, 65L80, 65L20.
Key words: Nonlinear delay-differential-algebraic equations with singular perturbation, Block boundary value methods, Unique solvability, Convergence, Global stability.

1. Introduction

In this paper, the numerical solution of the following nonlinear delay-differential-algebraic equations with singular perturbation (DDAESP) is investigated:

$$\begin{cases} \epsilon x'(t) = f(t, x(t), x(t-\tau), y(t), y(t-\tau)), & a \le t \le b, \quad 0 < \epsilon \ll 1, \\ y(t) = g(t, x(t), x(t-\tau), y(t), y(t-\tau)), & a \le t \le b, \\ x(t) = \varphi(t), \quad y(t) = \psi(t), & a - \tau \le t \le a, \end{cases} \tag{1.1}$$

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where ϵ is the singular perturbation parameter; $\tau > 0$ is the constant delay; $f : [a, b] \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$ and $g : [a, b] \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$ are sufficiently smooth functions subject to the following classic Lipschitz conditions with constants $L_j \geq 0, 1 \leq j \leq 8$. For all $t \in [a, b], u_1, u_2, v_1, v_2 \in \mathbb{R}^{d_1}$ and $w_1, w_2, r_1, r_2 \in \mathbb{R}^{d_2}$:

$$\begin{aligned}
&\|f(t, u_1, v_1, w_1, r_1) - f(t, u_2, v_2, w_2, r_2)\|_\infty \\
\leq &L_1 \|u_1 - u_2\|_\infty + L_2 \|v_1 - v_2\|_\infty + L_3 \|w_1 - w_2\|_\infty + L_4 \|r_1 - r_2\|_\infty , \qquad (1.2)\\
&\|g(t, u_1, v_1, w_1, r_1) - g(t, u_2, v_2, w_2, r_2)\|_\infty \\
\leq &L_5 \|u_1 - u_2\|_\infty + L_6 \|v_1 - v_2\|_\infty + L_7 \|w_1 - w_2\|_\infty + L_8 \|r_1 - r_2\|_\infty , \qquad (1.3)
\end{aligned}$$

where $\| \cdot \|$ is the corresponding maximum norm; $\varphi : [a - \tau, a] \to \mathbb{R}^{d_1}$ and $\psi : [a - \tau, a] \to \mathbb{R}^{d_2}$ are both the initial functions with the consistency condition:

$$\psi(a) = g(a, \varphi(a), \varphi(a - \tau), \psi(a), \psi(a - \tau)).$$

Hereafter, it is assumed that problem (1.1) has a unique exact solution $\{x(t), y(t)\}$. Especially, let $y(t) = \epsilon x'(t)$. Then, the following neutral singular perturbation problem with delay:

$$\begin{cases} \epsilon x'(t) = f(t, x(t), x(t - \tau), \epsilon x'(t - \tau)), & a \leq t \leq b, \\ x(t) = \varphi(t), & a - \tau \leq t \leq a, \end{cases} \tag{1.4}$$

is equivalent to

$$\begin{cases} \epsilon x'(t) = f(t, x(t), x(t - \tau), y(t - \tau)), & a \leq t \leq b, \\ y(t) = f(t, x(t), x(t - \tau), y(t - \tau)), & a \leq t \leq b, \\ x(t) = \varphi(t), \quad y(t) = \epsilon \varphi'(t), & a - \tau \leq t \leq a, \end{cases} \tag{1.5}$$

which is a special case of problem (1.1). Meanwhile, equation (1.1) is widely applied to many scientific fields, such as mechanics, automatics, physics, ecology, and so on [1, 2]. However, it is difficult to derive the exact solution of this equation, and various numerical solutions have been developed to replace the corresponding exact solution. The numerical simulation of this type of equations is crucial in practical applications, and it has attracted much attention from the computational mathematics community.

Due to the boundary layer effect of the solution, it is challenging to obtain a consistent layout of the optimal order of convergence [3]. To solve problem (1.1), the development of effective numerical methods becomes significant. The numerical studies of the singular perturbation problem originally begin from S.L. Campell's work [4, 5], where the stability of linear singularly perturbed delay differential equations was investigated. Then, Tian [6–8] studied the asymptotic expansion, exponential asymptotic stability, dissipativity, and exponential stability of nonlinear singular perturbation problems with delay (SPPDs). In the past years, various numerical methods have been applied to SPPDs, such as one-leg methods [9], linear multistep methods, Runge-Kutta methods [10, 11], circulant preconditioners [12], variational iteration methods [13], and parallel multistep hybrid methods [14]. In fact, Eq. (1.1) can more accurately describe the development of objective things than that without algebraic constraint. Thus, this type of singular perturbation problem with delay and algebraic constraint remains