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THE CONVERGENCE OF TRUNCATED EULER-MARUYAMA METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH PIECEWISE CONTINUOUS ARGUMENTS UNDER GENERALIZED ONE-SIDED LIPSCHITZ CONDITION*

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Abstract

In this paper, we consider the stochastic differential equations with piecewise continuous arguments (SDEPCAs) in which the drift coefficient satisfies the generalized one-sided Lipschitz condition and the diffusion coefficient satisfies the linear growth condition. Since the delay term $t-[t]$ of SDEPCAs is not continuous and differentiable, the variable substitution method is not suitable. To overcome this difficulty, we adopt new techniques to prove the boundedness of the exact solution and the numerical solution. It is proved that the truncated Euler-Maruyama method is strongly convergent to SDEPCAs in the sense of $L^{\bar{q}}(\bar{q}\geq 2)$. We obtain the convergence order with some additional conditions. An example is presented to illustrate the analytical theory.

Mathematics subject classification: 65L05, 65L60.
Key words: Stochastic differential equations, Piecewise continuous argument, One-sided Lipschitz condition, Truncated Euler-Maruyama method.

1. Introduction

In this paper, we consider the stochastic differential equations with piecewise continuous arguments (SDEPCAs)

$$dx(t)=f(x(t),x([t]))dt+g(x(t),x([t]))dB(t) \tag{1.1}$$

on $t\in[0,T]$ with the initial value $x(0)=x_0\in\mathbf{R}^n$, where $f:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$, $g:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^{n\times d}$ are measurable functions and $[t]$ denotes the greatest-integer part of t. SDEPCAs play an important role in biomedicine, physics, neural networks, control theory, etc, represent a hybrid of continuous and discrete dynamical systems and thus combine properties of both differential and difference equations [1–6]. SDEPCAs can be regarded as stochastic delay differential equation with variable delay $t-[t]$. However, $t-[t]$ is not continuous and differentiable. There have been many studies on SDEs and SDDEs, you can refer to [7–12].

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There have been many works on the numerical approximations of SDEPCAs such as [13–17] and [18]. The mean square convergence of the EulerMaruyama approximate solution is considered in [15] under the global Lipschitz condition and the linear growth condition. The strong convergence and exponential stability of the split-step method are considered for SDEPCAs with the one-sided Lipschitz condition for drift coefficient and the global Lipschitz condition for diffusion coefficient in [14] and with monotone condition plus polynomially growing conditions for drift coefficient and global Lipschitz condition for diffusion coefficient in [13]. The tamed Euler method is considered in [16] and [18]. Under the local Lipschitz and the general Khasminskii-type conditions, authors in [17] prove the convergence of explicit Euler method in probability for SDEPCAs.

Recently, the truncated Euler-Maruyama method is proposed for SDEs in [19] and [20]. Zhang, Song and Liu investigated the partially truncated Euler-Maruyama method for SDDEs in [21] and the truncated Euler-Maruyama(EM) method for stochastic functional differential equations in [22] with Khasminskii-type condition. Guo et al. in [23] studied the strong convergence of the truncated EM method for the stochastic differential equations with constant delay under the generalized Khasminskii-type condition such that

$$x^T f(x,y) + \frac{1}{2}|g(x,y)|^2 \le K_1(1+|x|^2+|y|^2) - K_2|x|^\beta + K_2|y|^\beta, \tag{1.2}$$

where $\beta > 2$. The authors show the strong convergence of the truncated Euler-Maruyama in qth moment for $q \in [1,2)$. Unfortunately, since $t-[t]$ is discontinuous and non-differentiable, the proof method in [23] is not applicable. According to the characteristics of $[t]$, on each time interval $[n, n+1)$, the SDEPCA is a SDE. When the coefficients satisfy the generalized one-sided Lipschitz condition, the key question is how to give the high-order moment estimation on the right side of the inequality. Therefore, we prove the boundedness of higher-order moments of the exact solution by estimating moments of solutions of a sequence of SDEs defined on successive intervals $[n, n+1)$. Besides, we prove that the truncated EM method strongly converges to SDEPCAs in the sense of $L^{\bar{q}}$ ($\bar{q} \ge 2$) and obtain the convergence order.

The rest of this paper is organized as follows. Section 2 introduces some basic assumptions and the properties of the exact solution. In Section 3, we construct the truncated Euler-Maruyama method. The pth moment boundedness and the convergence of the numerical solutions is presented in Section 4. Section 5 obtain the convergence order with some additional assumptions. An example is given to illustrate our conclusions in Section 6.

2. Theoretical Analysis for SDEPCAs

Throughout this paper, unless otherwise specified, we will use the following notations. If $\mathbf{A}$ is a vector or matrix, its transpose is denoted by $\mathbf{A}^{\mathrm{T}}$. If $x \in \mathbf{R}^n$, then $|\, x \,|$ is the Euclidean norm. If $\mathbf{A}$ is a matrix, we let $|\, \mathbf{A} \,| = \sqrt{trace(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$ be its trace norm. For two real numbers a and b, we use $a \vee b$ and $a \wedge b$ to denote $\max(a,b)$ and $\min(a,b)$, respectively. If D is a set, its indicator function is denoted by I_D. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be a d-dimensional Brownian motion defined on the probability space. Let C be a positive constant and its value may change between occurrences. In the following, we consider