Journal of Computational Mathematics
Vol.41, No.6, 2023, 1137–1170.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2207-m2022-0058

STABLE AND ROBUST RECOVERY OF APPROXIMATELY k-SPARSE SIGNALS WITH PARTIAL SUPPORT INFORMATION IN NOISE SETTINGS VIA WEIGHTED ℓ_p $(0 < p \leq 1)$ MINIMIZATION*

Biao Du and Anhua Wan[1)]
School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
Email: dubiao@mail2.sysu.edu.cn, wananhua@mail.sysu.edu.cn

Abstract

In the existing work, the recovery of strictly k-sparse signals with partial support information was derived in the ℓ_2 bounded noise setting. In this paper, the recovery of approximately k-sparse signals with partial support information in two noise settings is investigated via weighted ℓ_p $(0 < p \leq 1)$ minimization method. The restricted isometry constant (RIC) condition $\delta_{tk} < \frac{1}{p\eta^{\frac{2}{p}-1}+1}$ on the measurement matrix for some $t \in [1+\frac{2-p}{2+p}\sigma, 2]$ is proved to be sufficient to guarantee the stable and robust recovery of signals under sparsity defect in noisy cases. Herein, $\sigma \in [0,1]$ is a parameter related to the prior support information of the original signal, and $\eta \geq 0$ is determined by p, t and σ. The new results not only improve the recent work in [17], but also include the optimal results by weighted ℓ_1 minimization or by standard ℓ_p minimization as special cases.

Mathematics subject classification: 94A12, 94A15.
Key words: Signal recovery, weighted ℓ_p minimization, Approximately k-sparse signal, Noise setting, Reconstruction error bound, Restricted isometry property.

1. Introduction

As a data acquisition paradigm, compressed sensing has been a very active research area and has abundant applications [2,15,22]. Compressed sensing is particularly promising not only in applications such as hyperspectral imaging where taking measurements is costly, but also in applications such as medical and seismic imaging where the ambient dimension of the signal is very large [18].

In standard compressed sensing theory, one observes

$$y = Ax + z, \tag{1.1}$$

where $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ is an unknown sparse signal, $y \in \mathbb{R}^m$ is the observed signal, $A \in \mathbb{R}^{m\times n}$ is a measurement matrix with $m \ll n$, and $z \in \mathbb{R}^m$ denotes the noise in the measurement. One of the central goals of compressed sensing is to recover the original high-dimensional signal x based on the measurement matrix and the observed signal.

For signal recovery, the following noise settings

$$\mathcal{B}^{\ell_2}(\epsilon) := \left\{ z \in \mathbb{R}^m : \|z\|_2 \leq \epsilon \right\} \tag{1.2}$$

* Received March 12, 2022 / Revised version received May 23, 2022 / Accepted July 6, 2022 / Published online September 27, 2022 /
1) Corresponding author

and

$$\mathcal{B}^{DS}(\epsilon) := \left\{ z \in \mathbb{R}^m : \left\| A^T z \right\|_\infty \leq \epsilon \right\} \tag{1.3}$$

are of particular interest. Herein, $\epsilon \geq 0$ denotes some known margin. The ℓ_2 bounded noise setting (1.2) was considered for example in [14], and the *DS* noise setting (1.3) was motivated by the *Dantzig Selector* procedure in [5].

Denote the support of $x = (x_1, x_2, \ldots, x_n)^T$ as $\mathrm{supp}(x) = \{i : x_i \neq 0\}$. x is called k-sparse if the number of nonzero components in x is k at most, i.e., $\|x\|_0 = |\mathrm{supp}(x)| \leq k$.

The constrained $\ell_p\,(0 < p \leq 1)$ minimization method estimates the signal x by

$$\hat{x} = \arg\min_{x \in \mathbb{R}^n} \left\{ \|x\|_p^p : y - Ax \in \mathcal{B} \right\}, \tag{1.4}$$

where

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

is the ℓ_p (quasi-)norm of x and $\mathcal{B} \subseteq \mathbb{R}^m$ denotes some noise structure [21, 24, 28]. When in particular $p = 1$, the ℓ_p minimization model (1.4) becomes the standard ℓ_1 minimization model [1–3].

The following restricted isometry property (RIP) is a commonly used framework for sparse recovery.

Definition 1.1 ([4]). *Suppose $A \in \mathbb{R}^{m \times n}$ is a measurement matrix, k is an integer and $1 \leq k \leq n$. For the measurement matrix A, the restricted isometry constant (RIC) of order k is defined as the smallest number $\delta_k \geq 0$ such that for all k-sparse vectors $x \in \mathbb{R}^n$,*

$$(1 - \delta_k) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k) \|x\|_2^2 . \tag{1.5}$$

More generally, when k is not an integer, δ_k is defined as $\delta_{\lceil k \rceil}$, where $\lceil \cdot \rceil$ denotes the ceiling function [2].

In many practical applications, the original signal is not exactly k-sparse. As a consequence, the stable recovery of approximately sparse signals in noisy settings is of significant interest, and has been investigated under different sufficient RIC conditions by ℓ_p minimization model (1.4) [23, 26–28]. When $n \leq 4k$, under the assumption $p \in (0, \frac{3+2\sqrt{2}}{2}(1 - \delta_{2k})]$ for $\delta_{2k} \in (0, 1)$, Wen, Li and Zhu [26] proved the stable recovery of approximately k-sparse signals in the ℓ_2 bounded noise case. For $p \in (0, 1]$, Zhang and Li [28] derived the sharp condition

$$\delta_{2k} < \frac{\eta}{2 - p - \eta} \tag{1.6}$$

for the stable recovery of exactly k-sparse signals in noisy cases, where $\eta \in \left(1 - p, 1 - \frac{p}{2}\right)$ is the unique positive solution of the equation

$$\frac{p}{2} \eta^{\frac{2}{p}} + \eta - 1 + \frac{p}{2} = 0. \tag{1.7}$$

In our previous work [8, 24], general condition

$$\delta_{tk} < \delta^*(p, t) := \frac{\eta}{\dfrac{2 - p}{t - 1} - \eta} \tag{1.8}$$