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A MULTISCALE PROJECTION METHOD FOR SOLVING NONLINEAR INTEGRAL EQUATIONS UNDER THE LIPSCHITZ CONDITION*

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Abstract

We propose a multiscale projection method for the numerical solution of the irtatively regularized Gauss-Newton method of nonlinear integral equations. An a posteriori rule is suggested to choose the stopping index of iteration and the rates of convergence are also derived under the Lipschitz condition. Numerical results are presented to demonstrate the efficiency and accuracy of the proposed method.

Mathematics subject classification: 65J20, 65R20.
Key words: Nonlinear integral equations, Multiscale Galerkin method, parameter choice strategy, Gauss-Newton method.

1. Introduction

The aim of this paper is to propose a projection method of solving nonlinear integral equations of the type

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subset \mathbb{X} \to \mathbb{X}$ is a nonlinear Fredholm integral operater between the Hilert space $\mathbb{X}$ and defined by

$$F(x)(s) := \int_0^1 k(s,t,x(t))dt, \quad s \in [0,1],$$

where the kernel k is a continuous function on $[0,1] \times [0,1] \times R$. Eq. (1.1) is a typical example of an ill-posed problem, then the regularization technique has to be taken into account to yield the stable approximation [7,17,18].

Several regularization methods in the existing literature have been used to solve nonlinear integral equations. The regularization method [3,16,17], a two-step iterative process [15] have been considered to some extent and important results have already been obtained, but either lack of error analysis, or lack of an a posteriori rule.

Due to the faster convergence, the iteratively regularized Gauss-Newton method has received extensive attention in recent years [1,2,8,9]. Assume that the sequence $\{\alpha_k\}$ satisfy the following conditions:

$$\alpha_k > 0, \quad 1 \le \frac{\alpha_k}{\alpha_{k+1}} \le r, \quad \lim_{k\to\infty} \alpha_k = 0 \tag{1.2}$$

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with a constant $r > 1$, this method constructs the iterates $\{x_k^\delta\}$ by the following recursive algorithm:

$$x_{k+1}^\delta = x_k^\delta - (\alpha_k \mathcal{I} + F'(x_k^\delta)^* F'(x_k^\delta))^{-1}(F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) + \alpha_k(x_k^\delta - x_*)) \tag{1.3}$$

from the initial guess $x_0 = x^* \in D(F)$, where y^δ is the only available approximation of y satisfying

$$\|y^\delta - y\| \le \delta \tag{1.4}$$

with a given noise level $\delta > 0$.

We notice that most of the available results on the iteratively regularized Gauss-Newton are implemented in infinite dimensional space. In practical applications, we are more interested in considering this methods in a finite-dimensional setting.

There are many papers on the projection method to solve linear ill-posed equations, and many results have been obtained [5,10,11,14]. Therefore, we wonder if we can use the projection method to solve the nonlinear ill-posed problem? Motivated by this idea, this article attempts to solve the nonlinear ill-posed problem by using the projection method.

In this paper we propose a projection method for the iteratively regularized Gauss-Newton method and investigate the influence of the projection method. We focus on error analysis and and try to assert what conditions are appropriate for the discussions.

Throughout the paper it is assumed that F has continuous Fréchet derivative over $D(F)$. Assume that Eq. (1.1) has a solution $x^\dagger$ such that

$$B_\rho(x^\dagger) := \{x \in \mathbb{X}_n : \|x - x^\dagger\| \le \rho\} \subset D(F) \tag{1.5}$$

with a positive number $\rho > 10r\|x^* - x^\dagger\|$.

We next describe the multiscale Galerkin method for solving Eq. (1.3). We denote $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. We suppose that $\{\mathbb{X}_n, n \in \mathbb{N}_0\}$ is a sequence of finite dimensional subspaces of $\mathbb{X}$ satisfying [4]

$$\mathbb{X}_n \subset \mathbb{X}_{n+1}, n \in \mathbb{N}_0, \quad \overline{\bigcup_{n\in\mathbb{N}_0} \mathbb{X}_n} = \mathbb{X}.$$

For each $i \in \mathbb{N}$, let $\mathbb{W}_i$ be the orthogonal complement of $\mathbb{X}_{i-1}$ in $\mathbb{X}_i$. For a fixed $n \in \mathbb{N}$, we have the decomposition

$$\mathbb{X}_n = \mathbb{X}_0 \oplus^\perp \mathbb{W}_1 \oplus^\perp \cdots \oplus^\perp \mathbb{W}_n.$$

We assume that $\mathbb{W}_i$ has a basis $\{w_{ij}, j \in \mathbb{Z}_{w(i)}\}$. This means that $\mathbb{X}_n = \text{span}\{w_{ij} : (i,j) \in \mathbb{U}_n\}$, where $\mathbb{U}_n := \{(i,j) : j \in \mathbb{Z}_{w(i)}, i \in \mathbb{Z}_{n+1}\}$.

We now formulate the Galerkin method for solving Eq. (1.3). To this end, for each $n \in N_0$, we let $\mathcal{P}_n$ denote the orthogonal projection from $\mathbb{X}$ onto $\mathbb{X}_n$. The traditional Galerkin method for solving Eq. (1.3) is to find $x_{k,n}^\delta \in \mathbb{X}_n$ such that

$$\begin{cases} x_{0,n}^\delta = \mathcal{P}_n x^*, \\ x_{k+1,n}^\delta = x_{k,n}^\delta + (\alpha_k \mathcal{I} + \mathcal{A}_{k,n}^* \mathcal{A}_{k,n})^{-1}[\mathcal{A}_{k,n}^*(y^\delta - F(x_{k,n}^\delta)) + \alpha_k(\mathcal{P}_n x^* - x_{k,n}^\delta)], \end{cases} \tag{1.6}$$

where $\mathcal{A}_{k,n} := \mathcal{P}_n F'(x_{k,n}^\delta)\mathcal{P}_n$ and $\mathcal{A}_{k,n}^* := \mathcal{P}_n F'(x_{k,n}^\delta)^*\mathcal{P}_n$.

To write (1.6) in its equivalent matrix form, we make use of the multiscale basis functions. We write the solution $x_{k,n}^\delta \in \mathbb{X}_n$ as [14]

$$x_{k,n}^\delta = \sum_{(i,j)\in\mathbb{U}_n} c_{ij}^k w_{ij} \in \mathbb{X}_n.$$