Journal of Computational Mathematics
Vol.41, No.6, 2023, 1281–1304.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2205-m2021-0277

A DEEP LEARNING BASED DISCONTINUOUS GALERKIN METHOD FOR HYPERBOLIC EQUATIONS WITH DISCONTINUOUS SOLUTIONS AND RANDOM UNCERTAINTIES*

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Abstract

We propose a deep learning based discontinuous Galerkin method (D2GM) to solve hyperbolic equations with discontinuous solutions and random uncertainties. The main computational challenges for such problems include discontinuities of the solutions and the curse of dimensionality due to uncertainties. Deep learning techniques have been favored for high-dimensional problems but face difficulties when the solution is not smooth, thus have so far been mainly used for viscous hyperbolic system that admits only smooth solutions. We alleviate this difficulty by setting up the loss function using discrete shock capturing schemes–the discontinous Galerkin method as an example–since the solutions are smooth in the discrete space. The convergence of D2GM is established via the Lax equivalence theorem kind of argument. The high-dimensional random space is handled by the Monte-Carlo method. Such a setup makes the D2GM approximate high-dimensional functions over the random space with satisfactory accuracy at reasonable cost. The D2GM is found numerically to be first-order and second-order accurate for (stochastic) linear conservation law with smooth solutions using piecewise constant and piecewise linear basis functions, respectively. Numerous examples are given to verify the efficiency and the robustness of D2GM with the dimensionality of random variables up to 200 for (stochastic) linear conservation law and (stochastic) Burgers' equation.

Mathematics subject classification: 65C30, 65N99, 35L65.

Key words: Discontinuous Galerkin method, Loss function, Convergence analysis, Deep learning, Hyperbolic equation.

* Received October 1, 2021 / Revised version received January 26, 2022 / Accepted May 17, 2022 / Published online December 7, 2022 /

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1. Introduction

Hyperbolic equations with discontinuous solutions in the physical space arise in problems such as fluid mechanics, combustion, nonlinear acoustics, gas dynamics, and traffic flow [12,26]. One famous example is the compressible Euler equations in gas dynamics, which are the compressible Navier-Stokes equations without viscosity and heat conductivity. The inviscid equations develop discontinuous solutions, aka shocks, even if one starts from smooth initial data. Capturing shock waves has been an important subject in scientific computing and has been very successful [19,26]. Meanwhile, in reality, one may need to consider many sources of uncertainties that can arise in these models. They may be due to the incomplete knowledge of the model, such as the empirical equations of state or constitutive relations, imprecise measurement of physical parameters, and inaccurate measurement of boundary and initial data. Therefore, it is highly desirable to develop computational methods that not only capture the singular profile of solutions in the physical space but also take random uncertainties into account in the random space for high-fidelity simulations, along the line of uncertainty quantification (UQ) [23].

Due to the high dimensionality of the problems under study, it is natural to use deep-learning based approaches, which have been recently proposed for high-dimensional partial differential equations; see [13–15, 27–29, 31, 32, 35, 36] for examples and references therein. In these methods, the basic idea is to use a deep neural network (DNN) as the trail function to approximate the solution based on global optimization of a suitably chosen loss function. Specifically, the parameters in the DNN are optimized to make the DNN approximation satisfy the PDE and boundary/initial conditions as accurately as possible. Quite good approximate solutions are obtained for problems with dimensionality about 100. In all these methods, the loss function involves the (possibly higher-order) derivatives of the PDE solution, which prevents their ability to solve problems with discontinuous solutions, such as the (inviscid) Burgers' equation and the compressible Euler equations, and hence one usually solves viscous problems in which the solutions are smooth [31].

For hyperbolic equations with discontinuous solutions in the physical space, the discontinuous Galerkin (DG) method has been very popular [7–10, 25]. The flexibility of using discontinuous basis functions makes the DG methods capable of solving equations with discontinuous solutions, such as shock waves. There are many other shock capturing schemes [26] that can also be used. Here DG is chosen just as one example. For problems with uncertainties, the stochastic Galerkin (SG) method has been developed for PDEs with random coefficients [2,34], such as stochastic conservation laws [1, 24, 30], stochastic Hamilton–Jacobi equation [20] and stochastic wave equation [18, 33]. Compared with the Monte-Carlo (MC) method, the SG method achieves the spectral accuracy given the sufficient regularity of the PDE solution in the random space. Even though the SG methods are widely used for stochastic problems, their computational complexity grows exponentially with respect to the dimensionality of the random space. Therefore, when the dimensionality of the random space is large, the MC method is preferred.

In this work, we propose a deep learning based discontinuous Galerkin method (D2GM) to solve hyperbolic equations with discontinuous solutions and random uncertainties by combining the advantages of the DG method and DNNs. A key idea here is that at the discrete level, the DG method as an example here, the solution is smooth although its continuous counterpart is not. Thus one can expect that DNN will train better than the ones using AutoGrad in PyTorch or TensorFlow for time and/or spatial derivatives. We will give a convergence analysis for this