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UNCONDITIONAL CONVERGENCE AND ERROR ESTIMATES OF A FULLY DISCRETE FINITE ELEMENT METHOD FOR THE MICROPOLAR NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, we consider the initial-boundary value problem (IBVP) for the micropolar Naviers-Stokes equations (MNSE) and analyze a first order fully discrete mixed finite element scheme. We first establish some regularity results for the solution of MNSE, which seem to be not available in the literature. Next, we study a semi-implicit time-discrete scheme for the MNSE and prove $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the time discrete solution. Furthermore, certain regularity results for the time discrete solution are establishes rigorously. Based on these regularity results, we prove the unconditional $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the finite element solution of MNSE. Finally, some numerical examples are carried out to demonstrate both accuracy and efficiency of the fully discrete finite element scheme.

Mathematics subject classification: 76M10, 65M12, 65M15, 65M60.
Key words: Micropolar fluids, Regularity estimates, Euler semi-implicit scheme, Mixed finite element methods, Unconditional convergence.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded convex polyhedron domain. We consider the homogeneous incompressible viscous Newtonian Micropolar fluids. The microstructure systems consists of the incompressible Navier-Stokes equations for velocity with pressure and angular momentum equations for angular velocity, which are described by (see, e.g., [9, 10])

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\mu+\mu_r)\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} + 2\mu_r \mathbf{curl}\,\boldsymbol{w}, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - c_1\Delta\boldsymbol{w} - c_2\nabla\operatorname{div}\boldsymbol{w} + 4\mu_r\boldsymbol{w} = \boldsymbol{g} + 2\mu_r\mathbf{curl}\,\boldsymbol{u} \tag{1.3}$$

for $(x,t)\in\Omega\times(0,T)$, where $\boldsymbol{u}$ is the linear velocity; $\boldsymbol{w}$ is the angular velocity; p is the fluid pressure, $\boldsymbol{f}$ is the density of external body forces per unit mass, $\boldsymbol{g}$ is the body source of moments. c_1, c_2, μ_r and μ are material coefficients which are all constant coefficients greater than zero. The system (1.1)-(1.3) is supplemented with initial conditions for the linear velocity and the angular velocity

$$\boldsymbol{u}(x,0)=\boldsymbol{u}_0,\quad \boldsymbol{w}(x,0)=\boldsymbol{w}_0, \tag{1.4}$$

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together with the following the no-slip boundary condition for the linear velocity and the angular velocity:

$$\boldsymbol{u} = \mathbf{0}, \quad \boldsymbol{w} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T). \tag{1.5}$$

The system (1.1)-(1.5) describes the fluid particles may translate and rotate independently of the fluid (see, e.g., [4, 9, 10]). The Micropolar fluid models degenerates into the classical Navier-Stokes equations when microrotation is neglected ($\boldsymbol{w} = \mathbf{0}$) and $\mu_r = 0$. In the literature, many numerical methods have been developed for solving the nonstationary Navier-Stokes equations, see, e.g., [14,16,18,22,24] and the references therein. The microstructure fluid models is an important generalization of the incompressible Navier-Stokes equations. The derivation and physical discussion of the MNSE are referred to [3, 10, 21]. It is well konwn that the microstructure systems plays an important role in smart fluids and polarizable media [28,31]. In recent years, the micromachining technology has been used to develop a number of microfluidic systems, for example, silicon, glass, quartz and plastics [31]. Microchannels and chambers are the essential part of micropolar fluid system. Microchannels are also used for reactant transport, such as biochemical reaction chambers, physical particle separation, inkjet print heads, or as heat exchangers to cool computer chips. For more information about microchannels, we refer to [31] and the references therein.

The existence of solutions for the Micropolar fluid equations was established by Łukaszewicz [21]. In recent years, some attention has been paid to the numerical methods for this microstructure system. The penalty finite element method for the Micropolar fluid equations was proposed by Ortega *et al.* [30]. An important progress was made by Nochetto *et al.* [29], they proposed and analyzed an unconditionally stable semi-implicit fully discrete finite element scheme for MNSE with the nice features that the computation of the linear and angular velocities are decoupled. More recently, the convergence analysis of fractional time-stepping techniques for the Micropolar fluid equations had been obtained by Salgado [32]. The fractional time-stepping schemes or projection methods are known as an efficient decoupled schemes for incompressible flows, some extensions and applications of these methods to MNSE can be found in [20] with semi-discrete schemes and [35, 40] with fully discrete schemes. However, in our opinion, there are still some important problems in these references that remain to be solved. Firstly, all the convergence results for the finite element methods of the above works are obtained under proper regularity assumptions on the exact solution of the Micropolar fluid equations, which seems to be lacking in the literature. Secondly, due to the highly nonlinear structures of MNSE, the analysis of fully discrete finite element methods often requires that the time step τ satisfies an CFL like condition. For example, the condition $\tau h^{-1/2} \leq C$ are proposed for the error estimates in [29], where h is the spacial mesh width. Similar conditions are also imposed in the analysis in [32, 40]. Such smallness assumption on the time step will affect the applications of the finite element methods for MNSE.

The aim of this work is threefold. Firstly, we show certain regularity results for the solution of MNSE with the help of the energy method (see, e.g., [16, 36]). These regularity results are essential for the error estimates of numerical methods for the Micropolar fluid equations. Secondly, we will give the $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates for the Euler semi-implicit time discrete solution of MNSE. Furthermore, certain regularity results for the time discrete solution are also proved rigorously, which play a key role in the unconditional convergence analysis of fully discrete finite element method. However, for some of them, we have not seen a strict proof in relevant literature. Lastly, we consider the Euler semi-implicit fully discrete scheme based on mixed finite element method for the MNSE and prove the unconditional $\boldsymbol{L}^2$-$\boldsymbol{H}^1$ error estimates.