Journal of Computational Mathematics
Vol.42, No.2, 2024, 372–389.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2203-m2020-0192

BANDED M-MATRIX SPLITTING PRECONDITIONER FOR RIESZ SPACE FRACTIONAL REACTION-DISPERSION EQUATION*

Shiping Tang
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email: tangshp20@lzu.edu.cn
Aili Yang[1)]
School of Mathematics and Statistics, Hainan Normal University, Haikou 570000, China
Email: yangaili@lzu.edu.cn
Yujiang Wu
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email: myjaw@lzu.edu.cn

Abstract

Based on the Crank-Nicolson and the weighted and shifted Grünwald operators, we present an implicit difference scheme for the Riesz space fractional reaction-dispersion equations and also analyze the stability and the convergence of this implicit difference scheme. However, after estimating the condition number of the coefficient matrix of the discretized scheme, we find that this coefficient matrix is ill-conditioned when the spatial mesh-size is sufficiently small. To overcome this deficiency, we further develop an effective banded M-matrix splitting preconditioner for the coefficient matrix. Some properties of this preconditioner together with its preconditioning effect are discussed. Finally, Numerical examples are employed to test the robustness and the effectiveness of the proposed preconditioner.

Mathematics subject classification: 65N15, 65N30.
Key words: Riesz space fractional equations, Toeplitz matrix, conjugate gradient method, Incomplete Cholesky decomposition, Banded M-matrix splitting.

1. Introduction

We consider the following initial-boundary problem of Riesz space fractional reaction-dispersion equation (RSFRDE) [1]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -Ku(x,t) + K_\beta \dfrac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t), & x \in (a,b), \qquad t \in (0,T], \\ u(x,0) = \phi(x), & x \in (a,b), \\ u(x,t) = \psi(x,t), & x \in \mathbb{R}\backslash(a,b), \quad t \in [0,T], \end{cases} \tag{1.1}$$

where $1 < \beta < 2$ and the coefficients K, K_β are positive constants, $u(x,t)$ is an unknown function to be solved. In addition, $f(x,t)$ is the source term, and the Riesz space fractional

* Received July 20, 2020 / Revised version received February 14, 2022 / Accepted March 25, 2022 / Published online March 07, 2023 /
1) Corresponding author

operator $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ defined as

$$\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\Psi_\beta \left[{}_aD_x^\beta u(x,t) + {}_xD_b^\beta u(x,t) \right], \quad (x,t) \in (a,b) \times (0,T] \tag{1.2}$$

was used for describing anomalous diffusion. Here, the coefficient Ψ_β satisfies $\Psi_\beta = \frac{1}{2\cos(\pi\beta/2)} < 0$ for $\beta \in (1,2)$. When $K = 0$, problem (1.1) reduces to the Riesz fractional diffusion equation [2]. The definition of the left- and the right-sided Riemann-Liouville fractional derivatives ${}_aD_x^\beta$ and ${}_xD_b^\beta$ are given as [3, 4],

$$\begin{cases} {}_aD_x^\beta u(x,t) = \dfrac{1}{\Gamma(2-\beta)} \dfrac{\partial^2}{\partial x^2} \displaystyle\int_a^x \dfrac{u(\xi,t)}{(x-\xi)^{\beta-1}} d\xi, \\ {}_xD_b^\beta u(x,t) = \dfrac{1}{\Gamma(2-\beta)} \dfrac{\partial^2}{\partial x^2} \displaystyle\int_x^b \dfrac{u(\xi,t)}{(\xi-x)^{\beta-1}} d\xi, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function.

We should note that the boundary condition (BC) in (1.1) is defined on $\mathbb{R}\backslash(a,b)$ rather than $x = a, b$. This is because, take Lévy processes as an example, the paths of all proper Lévy processes, except Brownian motion with drift, are discontinuous, which means the boundary $x = a, b$ itself can not be hit by the majority of discontinuous sample trajectories. Therefore, we consider the generalized Dirichlet type BC on the domain $\mathbb{R}\backslash(a,b)$, where $\psi(x,t)$ satisfy that there exist positive M and C such that when $|x| > M$,

$$\frac{\psi(x,t)}{|x|^{\beta-\varepsilon}} < C \quad \text{for positive small } \varepsilon. \tag{1.3}$$

In particular, we choose $\psi(x,t) \equiv 0$ in this work, which is the so-called absorbing boundary condition [5].

In recent decades, due to the development of natural science, it has been found that, compared with classical integer order model, the fractional order model has huge advantages in many fields. The space-fractional diffusion equation (SFDE) has obtained wide attention, and it has been successfully used for explaining the anomalous dispersion phenomena in the real world, such as groundwater contaminant transport [6, 7], turbulent flow [8, 9], nonlocal heat conduction [10], biological systems [11], finance [12, 13], image processing [14] and so on. In addition, the SFDE was also used to model attenuation phenomena of acoustic waves in irregular porous random media, and acoustic wave propagation [15–17].

Since the fractional differential operator is nonlocal, it was shown that a native discretization of the SFDE, even though implicit, always leads to unstable numerical scheme. To overcome this difficulty, Meerschaet and Tadjeran [18, 19] proposed a shifted Grünwald discretization (SGD) to approximate SFDEs. Their method has been proved to be unconditionally stable. However, most of the available numerical methods, including SGD, for SFDEs tend to generate full coefficient matrices, which require computational cost of $\mathcal{O}(N^3)$ and storage of $\mathcal{O}(N^2)$ for solving the corresponding numerical scheme, where N is the number of grid points. Fortunately, Wang *et al.* [20] shown that the full coefficient matrix corresponding to the scheme possesses a special Toeplitz-like structure, which can be written as the sum of a scaled identity matrix and two diagonal-multiply-Toeplitz matrices. Hence, the storage requirement is significantly reduced from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$ and the complexity of the matrix-vector multiplication for the Toeplitz matrix is reduced from $\mathcal{O}(N^3)$ to $\mathcal{O}(N \log N)$ based on the fast Fourier transform [21–23].