Journal of Computational Mathematics
Vol.42, No.3, 2024, 662–678.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2205-m2021-0316

SHARP POINTWISE-IN-TIME ERROR ESTIMATE OF L1 SCHEME FOR NONLINEAR SUBDIFFUSION EQUATIONS*

Dongfang Li
School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China;
Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China
Email: dfli@hust.edu.cn
Hongyu Qin[1)] and Jiwei Zhang
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
Emails: qinhongyuhust@sina.com, jiweizhang@whu.edu.cn

Abstract

An essential feature of the subdiffusion equations with the α-order time fractional derivative is the weak singularity at the initial time. The weak regularity of the solution is usually characterized by a regularity parameter $\sigma \in (0,1) \cup (1,2)$. Under this general regularity assumption, we present a rigorous analysis for the truncation errors and develop a new tool to obtain the stability results, i.e., a refined discrete fractional-type Grönwall inequality (DFGI). After that, we obtain the pointwise-in-time error estimate of the widely used L1 scheme for nonlinear subdiffusion equations. The present results fill the gap on some interesting convergence results of L1 scheme on $\sigma \in (0,\alpha) \cup (\alpha,1) \cup (1,2]$. Numerical experiments are provided to demonstrate the effectiveness of our theoretical analysis.

Mathematics subject classification: 35R11, 65M12, 65M60.
Key words: Sharp pointwise-in-time error estimate, L1 scheme, Nonlinear subdiffusion equations, Non-smooth solutions.

1. Introduction

In this paper, we consider sharp pointwise-in-time error estimate of L1 scheme in time for solving the following nonlinear subdiffusion equations:

$$\partial_t^\alpha u - \Delta u = f(u), \quad x \in \Omega \times (0,T] \tag{1.1}$$

with the initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), && x \in \Omega, \\ &u(x,t) = 0, && x \in \partial\Omega \times [0,T], \end{aligned} \tag{1.2}$$

where $\Omega = (0,L)^d \subset \mathbb{R}^d$ $(d \geq 1)$. The time fractional Caputo derivative is defined as

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1. \tag{1.3}$$

* Received November 9, 2021 / Revised version received March 7, 2022 / Accepted May 6, 2022 / Published online April 24, 2023 /
1) Corresponding author

Here $\Gamma(\cdot)$ denotes the Gamma function. The equations provide a useful tool to describe anomalous diffusion in different physical situations, see, e.g., [4, 5, 21]. Hence, the theoretical and numerical analysis of the models have attracted the interest of plenty of researchers.

In developing numerical methods for solving the subdiffusion problem (1.1), an important consideration is that the solution u is typically less regular than in the case of a classical parabolic PDE (as the limiting case $\alpha \to 1$). For instance, Jin *et al.* [10] show that if the initial condition $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, the solution to problem (1.1) satisfies $\|\partial_t u(t)\|_{L^2(\Omega)} \le Ct^{\alpha-1}$. Maskari and Karaa [23] obtain that if $u_0 \in \dot{H}^\nu(\Omega)$ with $\nu \in (0,2]$, the solution of problem (1.1) satisfies $\|\partial_t u(t)\|_{L^2(\Omega)} \le Ct^{\nu\alpha/2-1}$, which implies that there exists a parameter $\sigma \in (0,\alpha]$ and $u_t \to \infty$ as $t \to 0^+$. One can refer to more works [1, 6, 12, 17, 18, 24, 25] on the discussion of the regularity of solutions. Without loss of generality, we assume the solution regularity satisfies

$$\|\partial_t^m u\|_{L^2(\Omega)} \le Ct^{\sigma-m} \quad \text{for} \quad m = 1, 2, \quad \sigma \in (0,1) \cup (1,2]. \tag{1.4}$$

Under the regularity assumption $\sigma = \alpha$ in (1.4), many works indicate that the convergence order with the maximum norm in time is $\mathcal{O}(\tau^\alpha)$, where τ is the temporal stepsize. One can refer to [11, 14, 19, 20, 28] for the L1 and L2-type schemes on the uniform meshes, [10, 22] for the convolution quadrature (CQ) Euler method and [8, 10] for the CQ BDF methods. In addition, numerical simulations show an interesting phenomenon that the convergence order of the L1 scheme on the uniform meshes is $\mathcal{O}(\tau^\alpha)$ as t tends to 0, and $\mathcal{O}(\tau)$ at the final time $t = T$. It motivates much works to consider the pointwise error estimate. For linear subdiffusion equations (i.e., $f(u) = 0$), Gracia *et al.* [3] proved the temporal error of L1 scheme on the uniform meshes is of $\tau t_n^{\alpha-1}$. Yan *et al.* [29] considered time-stepping error estimates of the modified L1 scheme. Jin *et al.* [7] showed if the initial condition $u_0(x) \in L_2(\Omega)$, the temporal error of L1 scheme on the uniform meshes is of τt_n^{-1}. After that, they [9] further obtained time-stepping error estimates of some high-order BDF convolution quadrature methods. Mustapha and McLean [26, 27] investigated time-stepping error bounds of discontinuous Galerkin methods for fractional diffusion problems.

For the nonlinear subdiffusion equations (1.1), Maskari and Karaa [23] studied the optimal pointwise-in-time error estimates based on the CQ Euler method. Kopteva [13] presented the pointwise-in-time error estimate of L1 scheme on the quasi-uniform temporal meshes for $\sigma = \alpha$ under appropriate conditions on the nonlinearity. The imposed conditions on the nonlinearity can guarantee the exact solutions have the upper and lower bounds, and a method of upper and lower solutions is introduced to address that the numerical solutions lie within a certain range similar to the exact solutions. Recently, the discrete fractional-type Grönwall inequality (DFGI) has received much attention and is well-studied, such as L1 scheme and a class of widely used schemes with fast algorithms [16, 19, 20] on nonuniform time steps, a general criteria [10] for L1 scheme and convolution quadrature generated by backward difference formulas on uniform time steps, and a Grüwald-Letnikov scheme [2, 30] on uniform time steps. As far as we know, these DFGIs have successfully proved the maximum error estimate, but cannot produce the pointwise error estimate.

The aims of this paper are to establish a refined DFGI, which is suitable to obtain the sharp pointwise error estimate of L1 scheme for problem (1.1) by taking the weak regularity of solutions into account. A fully discrete scheme is constructed by combining with the finite difference method for the spatial discretization, and a Newton linearized method for the nonlinear terms. Comparing with the stability analysis of continuous equations, the discrete integral kernel generally does not hold the semi-group property. It is the main reason why it is so dif-