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INVARIANTS-PRESERVING DU FORT-FRANKEL SCHEMES AND THEIR ANALYSES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH WAVE OPERATOR*

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Abstract

Du Fort-Frankel finite difference method (FDM) was firstly proposed for linear diffusion equations with periodic boundary conditions by Du Fort and Frankel in 1953. It is an explicit and unconditionally von Neumann stable scheme. However, there has been no research work on numerical solutions of nonlinear Schrödinger equations with wave operator by using Du Fort-Frankel-type finite difference methods (FDMs). In this study, a class of invariants-preserving Du Fort-Frankel-type FDMs are firstly proposed for one-dimensional (1D) and two-dimensional (2D) nonlinear Schrödinger equations with wave operator. By using the discrete energy method, it is shown that their solutions possess the discrete energy and mass conservative laws, and conditionally converge to exact solutions with an order of $\mathcal{O}(\tau^2 + h_x^2 + (\tau/h_x)^2)$ for 1D problem and an order of $\mathcal{O}(\tau^2 + h_x^2 + h_y^2 + (\tau/h_x)^2 + (\tau/h_y)^2)$ for 2D problem in H^1-norm. Here, τ denotes time-step size, while, h_x and h_y represent spatial meshsizes in x- and y-directions, respectively. Then, by introducing a stabilized term, a type of stabilized invariants-preserving Du Fort-Frankel-type FDMs are devised. They not only preserve the discrete energies and masses, but also own much better stability than original schemes. Finally, numerical results demonstrate the theoretical analyses.

Mathematics subject classification: 26A33, 34A08, 65M06, 65M12.
Key words: Nonlinear Schrödinger equations with wave operator, Du Fort-Frankel finite difference methods, Discrete energy and mass conservative laws, Numerical convergence.

1. Introduction

The nonlinear Schrödinger equations have been extensively applied in various mathematical and physical fields, such as, plasma physics, nonlinear optics and bimolecular dynamics. A nonlinear Schrödinger equation with wave operator, which was firstly proposed by Matsuuchi [31], was used to describe the nonlinear interactions between two waves travelling in opposite directions. In [14], the author studied the existence and uniqueness of the weak and strong solutions of the nonlinear Schrödinger equations with wave operator by means of Galerkin method, the regularity of the solutions, and the existence of the smooth solutions of 1D nonlinear Schrödinger equations with wave operator under weaker assumption. In [15], the author has devoted to researching the existence and nonexistence for this equation in the case of possessing different signs in nonlinear term under some conditions.

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In this paper, we consider the numerical solutions of the initial-boundary value problem for the nonlinear Schrödinger equations with wave operator as follows:

$$u_{tt} - \Delta u + i u_t + |u|^2 u + f(\mathbf{x}) u = 0, \quad \mathbf{x} \in \Omega, \quad t \in (0,T), \tag{1.1a}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad u_t(\mathbf{x},0) = u_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.1b}$$

$$u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in [0,T], \tag{1.1c}$$

by using invariants-preserving Du Fort-Frankel-type FDMs. Here $u(\mathbf{x},t)$ and $f(\mathbf{x})$ are an unknown complex function and a given real-valued function, respectively, $i = \sqrt{-1}$. For 1D case, we set $\Omega = (X_l, X_r)$ and denote $\mathbf{x} = x$. For 2D case, we write $\Omega = (X_l, X_r) \times (Y_l, Y_r)$ and $\mathbf{x} = (x,y)$. The conjugate complex number of u is denoted by $\overline{u}$.

Proposition 1.1 ([19, 32, 35]). *Let the mass and energy conservative laws for the problem* (1.1a)-(1.1c) *be defined as follows:*

$$Q(t) = 2\mathrm{Im}\langle u_t, u\rangle + \|u\|^2 + c, \tag{1.2a}$$

$$E(t) = \|u_t\|_{L^2}^2 + |u|_{H^1}^2 + \frac{1}{2}\|u\|_{L^4}^4 + \langle f, |u|^2\rangle + c, \tag{1.2b}$$

respectively. Then we have that $Q(t) = Q(0)$ and $E(t) = E(0)$. Here c is an arbitrary constant.

Proof. Acting the inner product of (1.1a) with $2u$, taking the imaginary part, applying Green formula and noting homogeneous Dirichlet boundary conditions, we obtain

$$\mathrm{Im}\langle u_{tt}, 2u\rangle + \mathrm{Re}\langle u_t, 2u\rangle = 0. \tag{1.3}$$

Besides, by simple computation, we have

$$\frac{d}{dt}(u_t \overline{u}) = u_{tt}\overline{u} + u_t \overline{u_t},$$

which shows that

$$\frac{d}{dt}\big[\mathrm{Im}(2u_t\overline{u})\big] = \mathrm{Im}(2u_{tt}\overline{u}). \tag{1.4}$$

Furthermore, by simple computation, we obtain

$$\frac{d}{dt}(u\overline{u}) = u_t\overline{u} + u\overline{u_t} = 2\mathrm{Re}(u_t\overline{u}). \tag{1.5}$$

Applying (1.4) and (1.5) to (1.3) shows that

$$\frac{d}{dt}Q(t) = 0, \quad Q(t) = Q(0). \tag{1.6}$$

Acting the inner product of (1.1a) with $2u_t$, applying the Green formula and noting homogeneous Dirichlet boundary conditions, taking the real part, using (1.5) and

$$\frac{d}{dt}(|u_t|^2) = 2\mathrm{Re}(u_{tt}\overline{u_t}), \quad -2\mathrm{Re}\langle \Delta u, 2u_t\rangle = \frac{d}{dt}|u|_{H^1}^2, \quad \frac{d}{dt}\left(\frac{1}{2}|u|^4\right) = 2|u|^2\mathrm{Re}(u\overline{u_t}), \tag{1.7}$$