Journal of Computational Mathematics
Vol.42, No.3, 2024, 885–910.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2211-m2021-0255

MIXED DISCONTINUOUS GALERKIN METHOD FOR QUASI-NEWTONIAN STOKES FLOWS*

Yanxia Qian
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Email: yxqian0520@163.com
Fei Wang[1)]
School of Mathematics and Statistics & State Key Laboratory of Multiphase Flow in Power Engineering, Xi'an Jiaotong University, Xi'an 710049, China
Email: feiwang.xjtu@xjtu.edu.cn
Wenjing Yan
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Email: wenjingyan@xjtu.edu.cn

Abstract

In this paper, we introduce and analyze an augmented mixed discontinuous Galerkin (MDG) method for a class of quasi-Newtonian Stokes flows. In the mixed formulation, the unknowns are strain rate, stress and velocity, which are approximated by a discontinuous piecewise polynomial triplet $\underline{\boldsymbol{\mathcal{P}}}_{k+1}^{\mathbb{S}}$-$\underline{\boldsymbol{\mathcal{P}}}_{k+1}^{\mathbb{S}}$-$\boldsymbol{\mathcal{P}}_k$ for $k \geq 0$. Here, the discontinuous piecewise polynomial function spaces for the field of strain rate and the stress field are designed to be symmetric. In addition, the pressure is easily recovered through simple postprocessing. For the benefit of the analysis, we enrich the MDG scheme with the constitutive equation relating the stress and the strain rate, so that the well-posedness of the augmented formulation is obtained by a nonlinear functional analysis. For $k \geq 0$, we get the optimal convergence order for the stress in broken $\underline{\boldsymbol{H}}(\mathbf{div})$-norm and velocity in $\boldsymbol{L}^2$-norm. Furthermore, the error estimates of the strain rate and the stress in $\underline{\boldsymbol{L}}^2$-norm, and the pressure in L^2-norm are optimal under certain conditions. Finally, several numerical examples are given to show the performance of the augmented MDG method and verify the theoretical results. Numerical evidence is provided to show that the orders of convergence are sharp.

Mathematics subject classification: 65N30, 65M60.
Key words: Quasi-Newtonian flows, Mixed discontinuous Galerkin method, Symmetric strain rate, Symmetric stress, Optimal convergence orders.

1. Introduction

The quasi-Newtonian Stokes equations arise in modeling flows of biological fluids, lubricants, paints, polymeric fluids, where the fluid viscosity is assumed to be a nonlinear function of the strain rate tensor [30, 35]. Let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^n$ with Lipschitz continuous boundary Γ. In this paper, we consider a class of Stokes equations whose viscosity depends nonlinearly on the strain rate, which is a characteristic feature of quasi-Newtonian flows: Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$, find a stress field $\underline{\boldsymbol{\sigma}}$, a velocity field $\boldsymbol{u}$

* Received September 3, 2021 / Revised version received April 6, 2022 / Accepted November 7, 2022 / Published online June 5, 2023 /
1) Corresponding author

and a pressure field p such that

$$\underline{\boldsymbol{\sigma}} = 2\mu(|\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u})|)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) - p\underline{\boldsymbol{I}} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{div}\underline{\boldsymbol{\sigma}} = -\boldsymbol{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\text{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma, \tag{1.1d}$$

$$\int_\Omega p d\boldsymbol{x} = 0, \tag{1.1e}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the nonlinear kinematic viscosity function of the fluid and $|\cdot|$ stands for Euclidean norm of tensors in $\mathbb{R}^{n\times n}$. Due to the incompressibility condition, we assume that $\boldsymbol{g}$ satisfies the compatibility condition $\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}ds = 0$, where $\boldsymbol{n}$ stands for the unit outward normal on Γ. Let us list some classic examples of the nonlinear kinematic viscosity μ for the quasi-Newtonian flows.

Power law:

$$\mu(t) = \mu_0 t^{\beta-2}, \quad \forall t \in \mathbb{R}^+ \quad \text{with} \quad \mu_0 > 0 \quad \text{and} \quad 1 < \beta < 2$$

serves to model the viscosity of many polymeric solutions and melts over a considerable range of shear rates [30].

Ladyzhenskaya law:

$$\mu(t) = (\mu_0 + \mu_1 t)^{\beta-2}, \quad \forall t \in \mathbb{R}^+ \quad \text{with} \quad \mu_0 \geq 0, \quad \mu_1 > 0 \quad \text{and} \quad \beta > 1$$

is used to model the fluids with large stresses [35].

Carreau law:

$$\mu(t) = \mu_0 + \mu_1(1+t^2)^{\frac{\beta-2}{2}}, \quad \forall t \in \mathbb{R}^+ \quad \text{with} \quad \mu_0 \geq 0, \quad \mu_1 > 0 \quad \text{and} \quad \beta \geq 1$$

is applied to model visco-plastic flows and creeping flow of metals [37].

The linear Stokes problem is recovered from the above laws when $\beta = 2$.

Many researchers aim at studying the efficient numerical methods for quasi-Newtonian flows and related problems, such as the conforming and nonconforming finite element method [4,6,17], the mixed finite element method [5, 21, 22], the dual-mixed finite element method [18, 31], the discontinuous Galerkin (DG) method [13, 16, 26, 27], the weak Galerkin method [43] and the virtual element method [14, 25] and so on. Traditionally, the numerical methods are studied based on the velocity-pressure variational formulation, where the velocity and the pressure are the main unknowns [8, 11]. Over the past decades, many researchers have paid attention to stress-based and pseudostress-based formulations [7, 15, 19, 23] because they provide a unified framework for both the Newtonian and non-Newtonian flows. Actually, a formulation comprising the stress as a fundamental unknown is unavoidable for non-Newtonian flows in which the constitutive law is nonlinear. Therefore, the mixed formulation is a good choice; besides the original unknowns, it yields direct approximations of several other physical interest quantities. For example, it is very desirable to calculate stress accurately and directly for flow problems involving interaction with solid structures. In [26], a pseudostress-based hybrid DG (HDG) scheme with BDM-like (Brezzi-Douglas-Marini) elements for the quasi-Newtonian Stokes flows was studied, and a priori error analysis was given. However, the $\underline{\boldsymbol{L}}^2$ error estimates of strain