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CONVERGENCE ANALYSIS OF NONCONFORMING QUADRILATERAL FINITE ELEMENT METHODS FOR NONLINEAR COUPLED SCHRÖDINGER-HELMHOLTZ EQUATIONS*

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Abstract

The focus of this paper is on two novel linearized Crank-Nicolson schemes with nonconforming quadrilateral finite element methods (FEMs) for the nonlinear coupled Schrödinger-Helmholtz equations. Optimal L^2 and H^1 estimates of orders $\mathcal{O}(h^2+\tau^2)$ and $\mathcal{O}(h+\tau^2)$ are derived respectively without any grid-ratio condition through the following two keys. One is that a time-discrete system is introduced to split the error into the temporal error and the spatial error, which leads to optimal temporal error estimates of order $\mathcal{O}(\tau^2)$ in L^2 and the broken H^1- norms, as well as the uniform boundness of numerical solutions in L^∞-norm. The other is that a novel projection is utilized, which can iron out the difficulty of the existence of the consistency errors. This leads to derive optimal spatial error estimates of orders $\mathcal{O}(h^2)$ in L^2-norm and $\mathcal{O}(h)$ in the broken H^1-norm under the H^2 regularity of the solutions for the time-discrete system. At last, two numerical examples are provided to confirm the theoretical analysis. Here, h is the subdivision parameter, and τ is the time step.

Mathematics subject classification: 65N15, 65N30.
Key words: Schrödinger-Helmholtz equations, Nonconforming FEMs, Linearized Crank-Nicolson scheme, Optimal error estimates.

1. Introduction

Consider the following generalized nonlinear coupled Schrödinger-Helmholtz equations:

$$\begin{cases} \mathrm{i}u_t + \Delta u + \phi f(|u|)u = 0, & (X,t) \in \Omega \times (0,T], \\ \alpha\phi - \beta^2 \Delta\phi = f(|u|)|u|^2, & (X,t) \in \Omega \times (0,T], \\ u = \phi = 0, & (X,t) \in \partial\Omega \times [0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

in which $X = (x,y), T < +\infty$ and Ω is a convex bounded domain in $\mathbb{R}^d\,(d=2)$ with the boundary $\partial\Omega$. $\mathrm{i} = \sqrt{-1}$, α, β are real nonnegative constants with $\alpha + \beta \neq 0$. $f : \mathbb{R} \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{C}$ are given functions. The complex-valued function u stands for the single particle wave function, the real-valued function $\phi(X,t)$ denotes the potential. The system (1.1) models many different physical phenomena in optics, quantum mechanics, and plasma physics, and so forth. When $\alpha = 0$, the system (1.1) reduces to the Schrödinger-Poisson

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model [13, 19, 28]. And when $\beta = 0$, the system (1.1) degenerates to a generalized nonlinear Schrödinger equation [1, 30]. Besides, we refer [5, 53] for other Schrödinger type equations such as the Schrödinger-Poisson-Slater model.

The nonlinear Schrödinger type equations have been attracted extensive attention from many researchers. A series of mathematical studies have been devoted for diverse Schrödinger type equations, such as well-posedness and dynamical properties, readers can refer to [1, 4, 7, 13, 19, 29] and the references therein. Along the numerical front, various numerical methods also have been investigated extensively for the nonlinear Schrödinger type equations, including finite difference methods [3, 6, 26, 44, 45, 52], spectral or pseudo-spectral methods [12, 14, 18, 46], FEMs [25,27,34,36,41,49,50], finite difference methods with the scalar auxiliary variable (SAV) formulation [9], Gauss collocation FEM based on the SAV approach [11], discontinuous Galerkin methods [16, 47], and other methods [2, 21, 24, 51]. Especially, the linearized backward Euler Galerkin FEMs [48], Crank-Nicolson Galerkin FEMs [42] and BDF2 Galerkin FEMs [38] were studied for Schrödinger-Helmholtz system. Both of them derived optimal L^2 error estimates for r-order conforming FEMs without any grid-ratio condition. Due to some pollution arising from the approximation used for the nonlinear terms $\phi f(|u|)u$ and $f(|u|)|u|^2$, only the error estimates at the time instant $t_{n+1/2}$ instead of the time division node t_n for the potential ϕ was derived in [42].

As we know, error estimates without any grid-ratio condition was first introduced by Li and Sun's work [23] through a time-space error splitting technique. Later on, this method has been established extensively to obtain optimal L^2 error estimates (see [20, 22, 34, 39, 41]) and to derive superconvergence error estimates (see [25, 33, 35, 43]) for various nonlinear problems. However, there are few results on the nonconforming FEMs by such method except for [35, 49]. Especially, how to extend the above conforming FEMs to nonconforming (quadrilateral) FEMs for the strongly nonlinear and coupled system such as Schrödinger-Helmholtz equations is a great challenge because several entitative difficulties need to be overcome. First, for the nonconforming FEM, it seems that we cannot directly obtain optimal L^2 error estimates, due to the existence of the consistency error. Although we can use superconvergent techniques to improve the convergence order, it requires higher regularity of solution [49], and need some special meshes, such as rectangular meshes. However, the regularity of the solution for the time discrete system is only $H^{2+\epsilon}$ $(0 \le \epsilon < 1)$ on polygonal area. Second, the inner product cannot exchange in complex space, i.e.,

$$\mathrm{Re}\big(\bar{\partial}_\tau \xi^n, \xi^n\big) = \frac{1}{2\tau}\big(\|\xi^n\|_0^2 - \|\xi^{n-1}\|_0^2 + \|\xi^n - \xi^{n-1}\|_0^2\big),$$

unlike in real space, we can use

$$(\bar{\partial}_\tau \xi^n, \xi^n) = \frac{1}{2\tau}(\|\xi^n\|_0^2 - \|\xi^{n-1}\|_0^2 + \|\xi^n - \xi^{n-1}\|_0^2),$$

directly. Third, the strong coupled nonlinear term $\phi f(|u|)u$ and strong nonlinear term $f(|u|)|u|^2$. These lead to the absence of $\bar{\partial}_\tau \xi^n$ on the left-hand of error equations when we take $v_h = \bar{\partial}_\tau \xi^n$ in (3.36a) to estimate $\|\xi^n\|_{1,h}$ directly. The so-called lifting techniques developed in [49] for Schrödinger equation and the skills utilized in [35] for parabolic equation does not work.

Up to now, we have not found nonconforming FEMs for the nonlinear Schrödinger-Helmholtz equations. Our goal in this work is to develop nonconforming FEMs for the system (1.1) on quadrilateral meshes, and then derive optimal L^2 and the broken H^1 estimates without any grid-ratio condition. Our work consists of the following three ingredients. First, different from