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DEGREE ELEVATION AND KNOT INSERTION FOR GENERALIZED BÉZIER SURFACES AND THEIR APPLICATION TO ISOGEOMETRIC ANALYSIS*

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Abstract

Generalized Bézier surfaces are a multi-sided generalization of classical tensor product Bézier surfaces with a simple control structure and inherit most of the appealing properties from Bézier surfaces. However, the original degree elevation changes the geometry of generalized Bézier surfaces such that it is undesirable in many applications, e.g. isogeometric analysis. In this paper, we propose an improved degree elevation algorithm for generalized Bézier surfaces preserving not only geometric consistency but also parametric consistency. Based on the knot insertion of B-splines, a novel knot insertion algorithm for generalized Bézier surfaces is also proposed. Then the proposed algorithms are employed to increase degrees of freedom for multi-sided computational domains parameterized by generalized Bézier surfaces in isogeometric analysis, corresponding to the traditional p-, h-, and k-refinements. Numerical examples demonstrate the effectiveness and superiority of our method.

Mathematics subject classification: 65D07, 65D17, 68U07.
Key words: Generalized Bézier surface, Degree elevation, Knot insertion, Isogeometric analysis, Refinement.

1. Introduction

The representation of surfaces, as one of the core research fields of computer-aided geometric design (CAGD) [7], has undergone tremendous advances in the past decades. Concerning geometric modeling, multiple powerful tools have been developed, such as Bézier surfaces and non-uniform rational B-spline (NURBS) surfaces [8,25]. Despite this advancement, the majority of these tools are inconvenient to represent n-sided ($n > 4$) surfaces required in some application scenarios. To overcome this problem, multi-sided surfaces, e.g. S-surfaces [21] and toric surfaces [16], are widely investigated [10, 18, 41, 42].

In 2016, Várady *et al.* [28] proposed a new polygonal parametric surface, named generalized Bézier (GB) surface, which is a generalization of the classical tensor product Bézier surface. The local coordinates of a GB surface are related to the generalized barycentric coordinates of the polygonal parametric domain. GB surfaces inherit the most of nice properties of Bézier surfaces. Based on [28], an enhanced version of GB surfaces was proposed in [29]. Salvi and Várady [30] proposed a new multi-sided surface scheme that permits domains with concave angles. Recently,

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Vaitkus *et al.* [27] introduce generalized B-spline (GBS) surfaces, which match B-spline surfaces with arbitrary geometric continuity. Compared with toric surfaces, GB surfaces are easier to generate high-quality parameterizations, which facilitates further analysis. In this paper, we focus on the first version of GB surfaces [28].

Although the GB surfaces is a powerful tool in multi-sided surface modeling, the degree elevation algorithm introduced in [28] may change the interior of the original surface. As the degree of a GB surface increases, the weight of the central control point increases, leading to the problem that the isoparametric curves move towards the central control point. The parameterization goes worse as a consequence. For this reason, we propose an improved degree elevation algorithm for GB surfaces that remains surfaces unchanged. Additionally, considering its capability of endowing local control, the locality plays a pivotal role in a surface. Based on the idea of deeming a Bézier surface as a B-spline surface with appropriate knot vectors, a novel knot insertion algorithm for GB surfaces is also proposed.

In 2005, Hughes *et al.* [5, 11] proposed the concept of isogeometric analysis (IGA), which has the potential for bridging the gap between finite element analysis (FEA) and computer-aid design (CAD). Given the boundary representation of a CAD model, constructing a spline-based mapping from its parametric domain to the computational domain is a crucial task in IGA. The quality of parameterization dramatically affects the accuracy and efficiency of the subsequent analysis [3, 26, 37]. To construct high-quality parameterizations, many methods have been proposed. Some methods are suitable for genus-0 domains, such as variational harmonic method [38] and Teichmüller mapping method [23]. For complex domains, especially for high-genus domains, single-patch parameterizations are not sufficient due to the topological flaw of general parametric surfaces. To this end, multi-patch configurations were extensively adopted [1, 36, 40]. Moreover, Lei *et al.* [17] proposed a novel automatic hexahedral mesh generation method, which lays a solid theoretic foundation for structured hex-meshing based on foliations. The above works mainly focus on isotropic parameterization methods that are independent of governing equations. With the same degrees of freedom (DOFs), anisotropic parameterizations customized for the governing equation may yield a more accurate numerical solution [14, 35]. Compared with the planar parameterization construction, the volumetric case is more challenging in both robustness and efficiency [13, 24, 39]. Xie *et al.* [33] handle volumetric modeling using interpolatory Catmull-Clark subdivision approaches. To improve computational efficiency, Xu *et al.* [34] proposed a framework for computation reuse in IGA. Recently, a deep-learning-based isogeometric analysis-reuse approach, called IGA-Reuse-Net, was proposed [32].

Though many methods are proposed, most of them focus on triangular or quadrilateral domains. However, triangular or quadrilateral representation is not suitable for multi-sided computational domains. As pointed out in [3, 18], using quadrilateral surfaces (e.g. Bézier and B-spline surfaces) to parametrize multi-sided domains may cause mesh degeneration or decreased continuity. As far as we know, only a few papers discuss the polygonal toric surface techniques in the parameterization for IGA [12, 18]. However, this method is sometimes unsatisfactory in terms of the quality of the parameterization. In this paper, GB surfaces are employed to IGA. And it is easier to construct satisfactory parameterizations.

Besides, there are three common methods in IGA to improve the accuracy of numerical solutions by increasing DOFs: knot insertion (h-refinement), degree elevation (p-refinement), and the combination of these two algorithms (k-refinement) [6,11]. By using the proposed degree elevation and knot insertion algorithms, we present the concepts of p-, h-, and k-refinements in