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MODIFIED SPLIT-STEP THETA METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION*

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Abstract

For solving the stochastic differential equations driven by fractional Brownian motion, we present the modified split-step theta method by combining truncated Euler-Maruyama method with split-step theta method. For the problem under a locally Lipschitz condition and a linear growth condition, we analyze the strong convergence and the exponential stability of the proposed method. Moreover, for the stochastic delay differential equations with locally Lipschitz drift condition and globally Lipschitz diffusion condition, we give the order of convergence. Finally, numerical experiments are done to confirm the theoretical conclusions.

Mathematics subject classification: 65L20.
Key words: Stochastic differential equation, Fractional Brownian motion, Split-step theta method, Strong convergence, Exponential stability.

1. Introduction

Recently, stochastic differential equations (SDEs) have been employed to describe many phenomena, such as finance [3], biomedical engineering [2, 12], water resources [16] and so on. Here, we consider the SDE

$$\begin{cases} \mathrm{d}x(t) = \mathscr{X}\big(x(t)\big)\mathrm{d}t + \mathscr{T}\big(x(t)\big)\mathrm{d}B(t), \quad t \in [0, \digamma], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $\mathscr{X}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ and $\mathscr{T}(\cdot) : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ are measurable functions, $B(\cdot)$ is an m-dimensional fractional Brownian motion (fBm) with the Hurst parameter $H \in (1/2, 1)$ and the initial value $x_0 \in \mathbb{R}^m$.

For most SDEs, it is always difficult to give the exact solutions. Thus, many scholars focus their attention on the numerical solutions. In the last decade, great progress has been made in the numerical analysis for SDEs driven by Brownian motion. The split-step theta (SST) scheme in [10] preserved exponential mean square stability for autonomous and non-autonomous equations under suitable conditions. The truncated Euler-Maruyama (EM) method in [13] was developed for solving SDEs with locally Lipschitz continuous coefficient. Later on, a series of truncated methods were studied, such as the truncated Milstein method [6], the multilevel Monte Carlo truncated EM method [7], the modified truncated EM method [11] and the full implicit truncated EM method [19].

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Compared with the Brownian motion, the lack of independent increments makes it difficult to deal with the fBm. As far as we know, the investigations of numerical methods for SDEs driven by fBm with locally Lipschitz coefficients have produced fewer results than those for SDEs driven by Brownian motion. Recently, [8] applied backward Euler scheme to the CIR problem driven by the fBm and obtained its strong convergence order. In [20], the authors constructed the implicit Euler scheme for the SDEs driven by fBm with locally Lipschitz drift and studied its strong convergence. To our best knowledge, there is few research on the SST method and the truncated EM method for the SDEs with fBm so far. Here, we will combine the SST method with the modified truncated EM method to provide a new modified split-step theta (MSST) method for solving the SDEs the fBm.

We finish this section by presenting its structure of the paper. Section 2 is concerned with some notations on the fBm and the Malliavin derivative, and give some necessary assumptions for the SDE (1.1). In Section 3, we propose the MSST method for this problem and obtain the convergence order. Section 4 analyzes the exponential stability in mean square of the proposed method. Section 5 studies the strong convergence of MSST method for stochastic delay differential equation (SDDE). Finally, the theoretical conclusions are demonstrated by two numerical experiments.

2. Preliminaries

Denote by $(\Omega, \Upsilon, \mathbb{P})$ a complete probability space, $\{\Upsilon_t\}_{t\geq 0}$ is increasing and continuous, $\{\Upsilon_0\}$ contains all $\mathbb{P}$-null sets. Unless otherwise specified, we always use the symbols below. The fBm has the continuous correction (see [1]), that is, for $n \geq 1$,

$$\mathbb{E}|B(w_1) - B(w_2)|^n = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)|w_1 - w_2|^{nH}, \quad \forall w_1, w_2 \in [0, \digamma]. \tag{2.1}$$

Let $L^2_\phi([0,\digamma])$ be an Hilbert space,

$$\phi(w_1, w_2) := H(2H-1)|w_1 - w_2|^{2H-2}$$

and $u : [0,\digamma] \to \mathbb{R}$ be a measurable function. Denote the random variable $\mathcal{F} : \Omega \to \mathbb{R}$ as $\mathcal{F} = \mathcal{K}(B(t_1), B(t_2), \cdots, B(t_n))$, where $\mathcal{K}$ is a smooth function with all bounded derivatives and $0 = t_0 < t_1 < \cdots < t_n = \digamma$. Define the Malliavin derivative (see [17])

$$\mathcal{D}_\varpi \mathcal{F} := \sum_{i=1}^{n} \frac{\partial \mathcal{K}}{\partial x_i}\big(B(t_1), B(t_2), \cdots, B(t_n)\big)1_{[0,t_i]}(\varpi), \quad \varpi \in [0,\digamma].$$

The space $\mathbb{D}^{1,p}$ is the completion of the set of all nonlinear functionals with

$$\|\mathcal{F}\|_{\mathbb{D}^{1,p}} := \big(\mathbb{E}[|\mathcal{F}|^p] + \mathbb{E}[\|\mathcal{D}\mathcal{F}\|^p_\phi]\big)^{\frac{1}{p}}.$$

Denote by δ the adjoint operator of derivative operator $\mathcal{D}$. If there is $\delta(\mathcal{G}) \in L^2_\phi([0,\digamma])$ such that $\mathbb{E}\left[\mathcal{F}\delta(\mathcal{G})\right] = \mathbb{E}\left[\langle \mathcal{G}, \mathcal{D}\mathcal{F}\rangle_\phi\right]$ for any $\mathcal{F} \in \mathbb{D}^{1,2}$, then $\mathcal{G}$ is integrable. Define the Skorohod integral

$$\int_0^{\digamma} \mathcal{G}(t)\delta B(t) := \delta(\mathcal{G}),$$