Journal of Computational Mathematics
Vol.42, No.5, 2024, 1277–1304.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2303-m2021-0309

A SIMPLE ITERATIVE ALGORITHM FOR MAXCUT*

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Abstract

We propose a simple iterative (SI) algorithm for the maxcut problem through fully using an equivalent continuous formulation. It does not need rounding at all and has advantages that all subproblems have explicit analytic solutions, the cut values are monotonically updated and the iteration points converge to a local optima in finite steps via an appropriate subgradient selection. Numerical experiments on G-set demonstrate the performance. In particular, the ratios between the best cut values achieved by SI and those by some advanced combinatorial algorithms in [Ann. Oper. Res., 248 (2017), 365–403] are at least 0.986 and can be further improved to at least 0.997 by a preliminary attempt to break out of local optima.

Mathematics subject classification: 90C27, 05C85, 65K10, 90C26, 90C32.
Key words: Maxcut, Iterative algorithm, Exact solution, Subgradient selection, Fractional programming.

1. Introduction

Given an undirected simple graph $G = (V, E)$ of order n with the vertex set V and the edge set E, a set pair (S, S') is called a cut of G if $S \cap S' = \varnothing$ and $S \cup S' = V$. The maxcut problem, one of Karp's 21 NP-complete problems [10], aims at finding a specific cut (S, S') of G to maximize the cut value

$$\mathrm{cut}(S) = \sum_{\{i,j\}\in E(S,S')} w_{ij}, \tag{1.1}$$

where $E(S, S')$ collects all edges cross between S and S' in E, and w_{ij} denotes the nonnegative weight on the edge $\{i, j\} \in E$.

Due to its widespread applications in various areas [1, 2, 4], several maxcut algorithms have been proposed to search for approximate solutions and usually fall into two distinct categories: discrete algorithms and continuous ones. The former mainly refer to the combinatorial algorithms for maxcut, which directly deal with the discrete objective function (1.1) and usually adopt both complicated techniques to break out of local optima and advanced heuristics improving the solution quality, such as the scatter search [12], the tabu search [15] and hybrid strategies within the framework of evolutionary algorithms [11, 13]. In contrast, the objective functions for the latter, often obtained from the relaxation of the discrete objective function

* Received November 2, 2021 / Revised version received March 1, 2022 / Accepted March 27, 2023 / Published online October 30, 2023 /
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(1.1), are continuous, and thus standard continuous optimization algorithms can be applied into the relaxed problems in a straightforward manner, for instance, the Goemans-Williamson (GW) algorithm [9], the CirCut algorithm resulted from the rank-two relaxation heuristics [3], the spectral cut (SC) algorithm [8, 16] and its recursive implementation (RSC) [5, 14, 18]. For all these continuous algorithms, rounding is essential and indispensable in obtaining a cut from a solution of the corresponding relaxed problem. The GW algorithm rounds the solution of a relaxing semidefinite programming via randomly selecting the hyperplanes until it achieves an expected cut value. A deterministic strategy, named Procedure-CUT, is adopted by CirCut to round the angle-vector solution to get a best possible associated cut. The SC algorithm obtains a cut by rounding the maximal eigenvector of graph Laplacian by a threshold, while the RSC algorithm recursively distributes part of unabsorbed points into two sets corresponding to a cut where the selection and assignment are determined by rounding the approximate solution of the dual Cheeger cut problem.

In this work, we propose a novel continuous algorithm for the maxcut problem, i.e. a simple iterative (SI) algorithm. Compared with the above-mentioned continuous maxcut algorithms, the proposed SI algorithm has the following distinct advantages. First, our inner subproblem can be solved analytically (see Theorem 3.1), whereas no matter the GW algorithm or the RSC algorithm needs call other optimization solvers for the inner subproblems. This constitutes the main reason why we use the adjunct word simple for the proposed algorithm. Second, our continuous optimization problem is directly equivalent to the maxcut problem (see Theorem 2.1), and the corresponding cut is updated in an iterative manner and converges to the local maximum (see Theorem 3.4). That is, the SI algorithm does not need any rounding at all. Finally, as an iterative algorithm, SI may select the cut by SC to be the initial point (see Section 4). In other words, it can also be used to improve the quality of the solution obtained from any other algorithms.

The rest is organized as follows. Section 2 establishes an equivalent continuous formulation of the maxcut problem (1.1) and a Dinkelbach-type iterative algorithm with global convergence. However, the solvability of the related inner subproblem can not be assured due to both the NP-hardness and the lack of convexity. To this end, in Section 3, we propose our simple iterative algorithm with an analytical solution to the inner problem. Both cost analysis and quality check are performed through numerical experiments on G-set in Section 4. Besides, in order to further improve the quality, a preliminary attempt to break out of local optima is also implemented there. We are concluded with a few remarks in Section 5.

2. Equivalent Continuous Problems

Given an undirected graph $G = (V, E)$ with nonnegative weights, let

$$I(\boldsymbol{x}) = \sum_{\{i,j\}\in E} w_{ij}|x_i - x_j|, \tag{2.1}$$

$$\|\boldsymbol{x}\|_\infty = \max\{|x_1|, \cdots, |x_n|\}, \tag{2.2}$$

$$F(\boldsymbol{x}) = \frac{I(\boldsymbol{x})}{\|\boldsymbol{x}\|_\infty}. \tag{2.3}$$

It can be readily verified that the nonnegative function $F(\boldsymbol{x})$ can achieve its maximum on $\mathbb{R}^n \setminus \{\boldsymbol{0}\}$ provided by its homogeneity of degree zero.