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A UNIFORM CONVERGENT PETROV-GALERKIN METHOD FOR A CLASS OF TURNING POINT PROBLEMS*

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Abstract

In this paper, we propose a numerical method for turning point problems in one dimension based on Petrov-Galerkin finite element method (PGFEM). We first give a priori estimate for the turning point problem with a single boundary turning point. Then we use PGFEM to solve it, where test functions are the solutions to piecewise approximate dual problems. We prove that our method has a first-order convergence rate in both L_h^∞ norm and a discrete energy norm when we select the exact solutions to dual problems as test functions. Numerical results show that our scheme is efficient for turning point problems with different types of singularities, and the convergency coincides with our theoretical results.

Mathematics subject classification: 65N06, 65B99.
Key words: Turning point problem, Petrov-Galerkin finite element method, Uniform convergency.

1. Introduction

Singularly perturbed problems have been widely studied in the fields of fluid mechanics, aerodynamics, convection-diffusion process, etc. In such problems, there exist boundary layers or interior layers because a small parameter is included in the coefficient of the highest derivative. Consider the following singularly perturbed turning point problem in one dimension:

$$\begin{cases} Lu = -\varepsilon u'' + p(x)u' + b(x)u = f(x), \quad x_L < x < x_R, \\ u(x_L) = u_L, \quad u(x_R) = u_R, \end{cases} \tag{1.1}$$

where $p(x)$ has zeros $z_1 < z_2 < \cdots < z_m$ on $[x_L, x_R]$. We assume $p, b,$ and f to be sufficiently smooth. Furthermore, we suppose

$$\begin{aligned} &b(x) - p'(x) \geq \gamma_0 > 0, \\ &b(x) \geq b_0 > 0 \end{aligned} \tag{1.2}$$

to ensure the well-posedness of the dual problems. Each zero of $p(x)$ is presumed to be a single root, i.e. $p'(z_i) \neq 0$.

From the asymptotic analysis, we know that there will be boundary/interior layers at some of z_i. Here we consider the following types of singularities:

(a) Exponential boundary layers (singularly perturbed problems without turning points).

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(b) Cusp-like interior layers (interior turning point problems).

(c) Boundary layers of other types (boundary turning point problems).

Singularly perturbed elliptic equations without turning points have been widely studied by researchers. Various numerical methods are utilized, where finite difference methods and finite element methods play prominent roles. El-Mistikawy and Werle [8] raise an exponential box scheme (EMW scheme) in order to solve Falkner-Skan equations. Kellogg *et al.* [2], Riordan and Stynes [28], etc., find this EMW scheme efficient when solving singularly perturbed elliptic equations. Fitted operator numerical methods, such as exponentially fitted finite difference method and Petrov-Galerkin method, are developed. Another class of methods, fitted mesh methods [6, 12, 16, 20, 31], show good adaptivity to different problems, while remeshing is necessary in some moving front problems.

A turning point problem is a class of equations in which the coefficient $p(x)$ vanishes at some points in the domain. Compared to singularly perturbed equations without turning points, interior layers and other types of boundary layers might appear in the solutions to turning point problems. O'Malley [24] and Abrahamson [1] analyze turning point problems in some common cases. Kellogg *et al.* [2] theoretically examine turning point problems with single interior turning points, and they use a modified EMW scheme, which obtains a first-order (or lower) convergence rate. Stynes and Riordan [28, 29] build a numerical scheme under Petrov-Galerkin framework and prove the uniform convergence in L^1 norm and L^∞ norm. Farrell [9] proposes sufficient conditions for an exponentially fitting difference scheme to be uniformly convergent for a turning point problem. Farrell and Gartland [10] modify the EMW scheme and construct a scheme with uniform first-order convergency, where parabolic cylinder functions are used in the computation. For other studies of turning point problems using fitted operator methods, please refer to [11, 21, 27, 34], for fitted mesh methods, please refer to [5, 19, 22, 25, 26, 30, 36].

We notice that most of the present research assume turning points to be away from the boundary. If a turning point meets an endpoint, the problem is called a boundary turning point problem, which has not been thoroughly studied. Vulanović [33] considers a turning point problem with an arbitrary single turning point and obtains uniform convergency using finite difference method on a non-equidistant mesh. Vulanović and Farrell [35] examine a multiple boundary turning point problem and make a priori estimates. However, estimates for single boundary turning point problems and numerical methods based on the uniform mesh are not given yet. In order to fill this blank, in this paper we estimate the derivatives of the solution to a standard single boundary turning point problem and raise an algorithm without particular mesh generation.

Petrov-Galerkin finite element method (PGFEM) is used in many problems. Dated back to 1979, Hemker and Groen [7] raise a method that treats problem (1.1) with Petrov-Galerkin method, where the coefficient $p(x)$ has a positive lower bound. The scheme in Farrell and Gartland [10] is based on the so-called patched function method, also interpreted as Petrov-Galerkin method. In references [3,4], Petrov-Galerkin method and discontinuous Petrov-Galerkin method are implemented in elliptic equations in two dimensions, demonstrating their efficiency and convergency.

Tailored finite point method (TFPM) is raised by Han *et al.* [15], which is designed to solve PDEs using properties of the solutions, especially for singularly perturbed problems. TFPM could handle exponential singularities well, while simple difference methods might sometimes