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VARIABLE STEP-SIZE BDF3 METHOD FOR ALLEN-CAHN EQUATION*

Minghua Chen[1], Fan Yu and Qingdong Zhang
School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, China
Emails: chenmh@lzu.edu.cn, yuf20@lzu.edu.cn, zhangqd19@lzu.edu.cn
Zhimin Zhang
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
Email: zmzhang@csrc.ac.cn

Abstract

In this work, we analyze the three-step backward differentiation formula (BDF3) method for solving the Allen-Cahn equation on variable grids. For BDF2 method, the discrete orthogonal convolution (DOC) kernels are positive, the stability and convergence analysis are well established in [Liao and Zhang, Math. Comp., 90 (2021), 1207–1226] and [Chen, Yu, and Zhang, arXiv:2108.02910, 2021]. However, the numerical analysis for BDF3 method with variable steps seems to be highly nontrivial due to the additional degrees of freedom and the non-positivity of DOC kernels. By developing a novel spectral norm inequality, the unconditional stability and convergence are rigorously proved under the updated step ratio restriction $r_k := \tau_k/\tau_{k-1} \leq 1.405$ for BDF3 method. Finally, numerical experiments are performed to illustrate the theoretical results. To the best of our knowledge, this is the first theoretical analysis of variable steps BDF3 method for the Allen-Cahn equation.

Mathematics subject classification: 65L06, 65M12.
Key words: Variable step-size BDF3 method, Allen-Cahn equation, Spectral norm inequality, Stability and convergence analysis.

1. Introduction

The objective of this paper is to present a rigorous stability and convergence analysis of the BDF3 method with variable steps for solving the Allen-Cahn equation [2]

$$\begin{cases} \partial_t u - \varepsilon^2 \Delta u + f(u) = 0, \quad (x,t) \in \Omega \times (0,T], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where the nonlinear bulk force is given by $f(u) = F'(u) = u^3 - u$, and the parameter $\varepsilon > 0$ represents the interface width. For simplicity, we consider the periodic boundary conditions. The above Allen-Cahn equation can be viewed as an L^2-gradient flow of the following free energy functional:

$$E[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right) dx, \quad F(u) = \frac{1}{4}(u^2-1)^2. \tag{1.2}$$

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1) Corresponding author

In other words, the Allen-Cahn equation (1.1) admits the energy dissipation law

$$\frac{dE[u]}{dt} = -\int_\Omega |\partial_t u|^2 dx \le 0. \tag{1.3}$$

Let $N \in \mathbb{N}$ and choose the nonuniform time levels $0 = t_0 < t_1 < \cdots < t_N = T$ with the time-step $\tau_k = t_k - t_{k-1}$ for $1 \le k \le N$. For any time sequence $\{v^n\}_{n=0}^N$, denote

$$\nabla_\tau v^n := v^n - v^{n-1}, \quad \partial_\tau v^n := \frac{\nabla_\tau v^n}{\tau_n}, \quad n \ge 1.$$

For $k = 1, 2, 3$, let $\Pi_{n,k} v$ denote the Lagrange interpolating polynomial of a function v over $k+1$ nodes $t_n, t_{n-1}, \dots, t_{n-k}$. Define the adjacent time step ratio

$$r_k := \frac{\tau_k}{\tau_{k-1}}, \quad k \ge 2.$$

Let $v^n = v(t_n)$. The BDF3 scheme is defined by [5, 13, 15, 16, 21, 23]

$$\begin{aligned} D_3 v^n = (\Pi_{n,3} v)'(t_n) &= b_0^{(n)} \nabla_\tau v^n + b_1^{(n)} \nabla_\tau v^{n-1} + b_2^{(n)} \nabla_\tau v^{n-2} \\ &= \sum_{k=1}^n b_{n-k}^{(n)} \nabla_\tau v^k, \quad n \ge 3, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} b_0^{(n)} &= \frac{(1 + r_{n-1})\big[1 + 2r_n + r_{n-1}\big(1 + 4r_n + 3r_n^2\big)\big]}{\tau_n(1 + r_n)(1 + r_{n-1})(1 + r_{n-1} + r_n r_{n-1})}, \\ b_1^{(n)} &= -\frac{r_n^2[(1 + 2r_{n-1} + r_n r_{n-1})^2 - r_{n-1}(1 + r_{n-1})]}{\tau_n(1 + r_n)(1 + r_{n-1})(1 + r_{n-1} + r_n r_{n-1})}, \\ b_2^{(n)} &= \frac{r_n^2 r_{n-1}^3 (1 + r_n)^2}{\tau_n(1 + r_n)(1 + r_{n-1})(1 + r_{n-1} + r_n r_{n-1})}, \quad b_j^{(n)} = 0, \quad j \ge 3. \end{aligned} \tag{1.5}$$

Since BDF3 scheme needs three starting values, for concreteness, we use BDF1 and BDF2 schemes to respectively compute the first-level solution u^1 and second-level solution u^2, namely,

$$D_3 v^1 := D_1 v^1 = \frac{\nabla_\tau v^1}{\tau_1}, \quad D_3 v^2 := D_2 v^2 = \frac{1 + 2r_2}{\tau_2(1 + r_2)} \nabla_\tau v^2 - \frac{r_2^2}{\tau_2(1 + r_2)} \nabla_\tau v^1. \tag{1.6}$$

We recursively define a sequence of approximations u^n to the nodal values $u(t_n)$ of the exact solution by BDF3 method

$$D_3 u^n - \varepsilon^2 \Delta u^n + f(u^n) = 0, \quad n \ge 1, \tag{1.7}$$

where the initial data $u^0 = u_0$ and $f(u^n) = (u^n)^3 - u^n$.

The BDF3 operator (1.4) and (1.6) are regarded as a discrete convolution summation

$$D_3 v^n = \sum_{k=1}^n b_{n-k}^{(n)} \nabla_\tau v^k, \quad n \ge 1, \tag{1.8}$$

where

$$b_0^{(1)} = \frac{1}{\tau_1}, \quad b_0^{(2)} = \frac{1 + 2r_2}{\tau_2(1 + r_2)}, \quad b_1^{(2)} = -\frac{r_2^2}{\tau_2(1 + r_2)}$$

in (1.6) and $b_{n-k}^{(n)}$ in (1.5).