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NUMERICAL METHODS FOR APPROXIMATING STOCHASTIC SEMILINEAR TIME-FRACTIONAL RAYLEIGH-STOKES EQUATIONS*

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Abstract

This paper investigates a semilinear stochastic fractional Rayleigh-Stokes equation featuring a Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ in time and a fractional time-integral noise. The study begins with an examination of the solution's existence, uniqueness, and regularity. The spatial discretization is then carried out using a finite element method, and the error estimate is analyzed. A convolution quadrature method generated by the backward Euler method is employed for the time discretization resulting in a fully discrete scheme. The error estimate for the fully discrete solution is considered based on the regularity of the solution, and a strong convergence rate is established. The paper concludes with numerical tests to validate the theoretical findings.

Mathematics subject classification: 65M60, 65M12, 65M15.
Key words: Riemann-Liouville fractional derivative, Stochastic Rayleigh-Stokes equation, Finite element method, Convolution quadrature, Error estimates.

1. Introduction

We investigate the following semilinear stochastic fractional order Rayleigh-Stokes problem:

$$u_t(t,x) + \big(1+\partial_t^{1-\alpha}\big)Au(t,x) = f\big(u(t,x)\big) + \partial_t^{-\gamma}\dot{W}(t), \quad t\in(0,T], \quad x\in\mathcal{D}, \tag{1.1a}$$

$$u(t,x) = 0, \quad t\in(0,T], \quad x\in\partial\mathcal{D}, \tag{1.1b}$$

$$u(0,x) = u_0, \quad x\in\mathcal{D}, \tag{1.1c}$$

where $0<\alpha<1$, $0\le\gamma\le 1$ and $T>0$ is a fixed time. Here, $A=-\Delta$ denotes the negative Laplace operator with its domain $D(A)=H^2(\mathcal{D})\cap H_0^1(\mathcal{D})$ and $\mathcal{D}\subset\mathbb{R}^d, d\le 3$, is an open convex polygonal domain with a boundary $\partial\mathcal{D}$. The operator $\partial_t^{-\gamma}$ denotes the Riemann-Liouville time-fractional integral operator defined by

$$\partial_t^{-\gamma}\varphi(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\varphi(s)\,ds,$$

where $\Gamma(\cdot)$ is the usual Gamma function. The operator $\partial_t^{1-\alpha} := \partial_t\partial_t^{-\alpha}$ denotes the Riemann-Liouville time-fractional derivative, where $\partial_t=\partial/\partial t$. In the model (1.1), the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the global Lipschitz condition

$$|f(t)-f(s)| \le L|t-s|, \quad \forall\, t,s\in\mathbb{R}, \quad L>0. \tag{1.2}$$

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The noise $\{W(t)\}_{t\geq 0}$ is an $L^2(\mathcal{D})$-valued Wiener process with a covariance operator Q with respect to a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, $\dot{W}(t) := dW(t)/dt$ is its formal derivative. The initial data u_0 is an $\mathcal{F}_0$-measurable random variable with values in $L^2(\mathcal{D})$.

The stochastic Rayleigh-Stokes problem is a model used to describe the dynamic behavior of non-Newtonian fluids, where the time-fractional derivative $\partial_t^{1-\alpha}$ is utilized to capture fluid elasticity (as noted in [1, 4, 5] and related references). The numerical approximation of linear time-fractional stochastic evolution equations has been extensively studied, with several works including [7, 9, 10, 13, 14, 17]. Jin *et al.* [10] analyzed the strong and weak convergence of a numerical scheme for subdiffusion equations with fractionally integrated Gaussian noise, which was created using the Galerkin finite element method for the spatial aspect and convolution quadrature for the fractional derivative. In [17], the focus was on a stochastic subdiffusion problem driven by integrated space-time white noise, with the L1 scheme and Lubich's first order convolution quadrature formula being used to approximate the time-fractional derivative and time-fractional integral, respectively. The study established a strong convergence rate.

The numerical analysis of semilinear time-fractional stochastic equations has been explored in recent works such as [3,11]. Kang *et al.* [11] investigated a stochastic space and time-fractional subdiffusion problem that included a fractionally integrated additive noise and a globally Lipschitz term $f(u)$. The authors regularized the problem and derived error estimates based on the properties of the Mittag-Leffler functions. More recently, in [3], the authors studied a stochastic time-fractional Allen-Cahn model perturbed by a fractionally integrated Gaussian noise. The Galerkin finite element method was used for the spatial approximation, and a convolution quadrature was used to approximate both the fractional derivative and integral. By utilizing the temporal Hölder continuity property of the solution, strong convergence rates for the error were derived. In both [3, 11], conditions on α and γ were imposed for the well-posedness of the stochastic time-fractional models.

In this study, the solution is represented in an integral form and global existence and uniqueness of solution are discussed. The regularity of the solution in both space and time is established. The main objective of this work is to prove a strong convergence rate in $L^2(\Omega; L^2(\mathcal{D}))$ for the fully discrete scheme using a semigroup type approach. The spatial discretization is performed using a Galerkin finite element method, while the noise is approximated by an L^2-projection. Under the condition $-\alpha(2-r)/2+\gamma > -1/2$, where r is defined in (4.6), we derive error estimates for the semidiscrete scheme. The fully discrete scheme is then obtained by applying a convolution quadrature generated by the backward Euler method for the fractional derivative and integral. By exploiting the solution regularity and the globally Lipschitz property of the source term f given in (1.2), the error estimate is analyzed and a strong convergence rate for the fully discrete scheme is proved.

The paper is structured as follows. In Section 2, we introduce notations and recall some properties of Wiener processes. In Section 3, the representation of the solution is discussed along with its existence, uniqueness, and regularity. Section 4 deals with the spatial discretization and the error analysis of the resulting semi-discrete scheme. In Section 5, error estimates for the fully discrete scheme are established. Finally, in Section 6, numerical experiments are conducted to validate the theoretical results.

Throughout the paper, we use c and C to denote generic constants that may change from one occurrence to another, but are always independent of the mesh size h and time step size τ. Additionally, we simplify the notation by writing $u(t)$ instead of $u(t, x)$.