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CRANK-NICOLSON GALERKIN APPROXIMATIONS FOR LOGARITHMIC KLEIN-GORDON EQUATION*

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Abstract

This paper presents three regularized models for the logarithmic Klein-Gordon equation. By using a modified Crank-Nicolson method in time and the Galerkin finite element method (FEM) in space, a fully implicit energy-conservative numerical scheme is constructed for the local energy regularized model that is regarded as the best one among the three regularized models. Then, the cut-off function technique and the time-space error splitting technique are innovatively combined to rigorously analyze the unconditionally optimal and high-accuracy convergence results of the numerical scheme without any coupling condition between the temporal step size and the spatial mesh width. The theoretical framework is uniform for the other two regularized models. Finally, numerical experiments are provided to verify our theoretical results. The analytical techniques in this work are not limited in the FEM, and can be directly extended into other numerical methods. More importantly, this work closes the gap for the unconditional error/stability analysis of the numerical methods for the logarithmic systems in higher dimensional spaces.

Mathematics subject classification: 65N30, 65N06, 65N12.
Key words: Logarithmic Klein-Gordon equation, Finite element method, Cut-off, Error splitting technique, Convergence.

1. Introduction

In this paper, we consider the Klein-Gordon equation with the logarithmic nonlinear term (LogKGE)

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + \lambda u(\mathbf{x},t) f\big(|u(\mathbf{x},t)|^2\big) = 0, \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{1.1a}$$

$$u(\mathbf{x},0) = \phi_0(\mathbf{x}), \quad u_t(\mathbf{x},0) = \phi_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.1b}$$

$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times [0,T], \tag{1.1c}$$

where $u(\mathbf{x},t)$ is a real valued scalar field, λ is a parameter measuring the force of the nonlinear interaction, $\phi_0(\mathbf{x})$ and $\phi_1(\mathbf{x})$ are given sufficiently smooth functions, $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a bounded convex polygonal or polyhedral domain fixed on a Lipschitz continuous boundary $\partial\Omega$, and

$$f(\rho) = \ln\rho, \quad \rho = |u(\mathbf{x},t)|^2 > 0. \tag{1.2}$$

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The LogKGE (1.1) admits the law of energy conservation defined by

$$E(t) = \int_\Omega \left(|u_t(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2 + |u(\mathbf{x},t)|^2 + \lambda F\left(|u(\mathbf{x},t)|^2\right)\right) d\mathbf{x} \equiv E(0), \quad t \in [0,T], \tag{1.3}$$

where $u(\cdot,t) \in H^1(\mathbb{R}^d), u_t(\cdot,t) \in L^2(\mathbb{R}^d)$ and

$$F(\rho) = \int_0^\rho f(s)ds = \int_0^\rho \ln s ds = \rho \ln \rho - \rho, \quad \rho > 0. \tag{1.4}$$

The logarithmic nonlinearity is widely used in various physical models for different fields of research, such as the logarithmic Schrödinger equation (LogSE) established in quantum mechanics or quantum optics [15, 16], the logarithmic Korteweg-de Vries equation and logarithmic Kadomtsev-Petviashvili equation applied to characterize oceanography and fluid dynamics [22, 45], the Cahn-Hilliard equation with logarithmic potentials [18, 20] studied in material sciences, and so on. Additionally, the LogKGE is regarded as the relativistic version of the LogSE [13], which has been introduced into the quantum field theory by Rosen [37]. This equation has attracted widespread attention due to its fundamental importance in the study of quantum field theory and its connection to various physical phenomena.

In the past decades, many scholars have devoted themselves to studying the well-posedness of the Cauchy problem for LogKGEs. Bartkowski *et al.* [13] proved the existence and uniqueness of weak solutions and classical solutions for one-dimensional LogKGE. Later, Natali *et al.* [35] gave the orbital stability results of periodic standing waves of one-dimensional LogKGE. By employing the auxiliary equation method, Alzaleq *et al.* [2] found new bounded and unbounded exact traveling wave solutions for LogKGE with three different forms. In [46], the author indicated that LogKGE possessed Gaussons: Solitary wave solutions of Gaussian shape. Since the analytical solutions of most nonlinear Klein-Gordon equations are not easy to find, a series of numerical methods have been considered, including finite difference methods (FDMs) [8, 11, 12, 14, 30, 50], FEMs [17, 24, 44], spectral methods [9], exponential wave integrator [8] and operator splitting [10] Fourier pseudospectral methods, and so on. However, due to the blow-up of the logarithmic nonlinear term near the origin, these numerical methods cannot be directly applied to logarithmic equations.

In order to avoid the blow-up, Bao *et al.* [5,6] proposed a regularized FDM and a regularized splitting method for LogSE, and established their error bound. Li *et al.* [27] applied the FDM to solve the numerical solutions of the regularized LogSE in an unbounded domain. Later, for the LogKGE, two energy-conservative regularized FDMs were employed and their error estimates were obtained [48, 49]. It is well known that logarithmic function will only appear numerical blow-up when $\rho \to 0^+$, and this phenomenon will not occur when the value of ρ is large. Therefore, Bao *et al.* [7] recently presented an energy regularized logarithmic Schrödinger equation (ERLogSE) through local energy regularization (LER) technique, that is, a sequence of polynomials approximation to the interaction energy density $F(\rho)$ at near origin. Inspired by above works, Yan *et al.* [47] extended the LER technique to the LogKGE. A conservative Crank-Nicolson FDM and an explicit FDM were raised for the obtained ERLogKGE. Through the above analysis and our knowledge, it is found that there exists no research focusing on the FEM for the LogKGE. However, we must emphasize that the finite element discretization allows us to work in a very low regularity states, which cannot be done by spectral methods or FDMs. Additionally, the FEM exhibits excellent adaptability to complex geometric regions and boundary conditions. In this work, we aim to bridge this gap by developing an energy-conservative FEM for the LogKGE (1.1).