Journal of Computational Mathematics
Vol.43, No.3, 2025, 673–689.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2305-m2023-0088

AN ITERATIVE TWO-GRID METHOD FOR STRONGLY NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS*

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Abstract

We design and analyze an iterative two-grid algorithm for the finite element discretizations of strongly nonlinear elliptic boundary value problems in this paper. We propose an iterative two-grid algorithm, in which a nonlinear problem is first solved on the coarse space, and then a symmetric positive definite problem is solved on the fine space. The main contribution in this paper is to establish a first convergence analysis, which requires dealing with four coupled error estimates, for the iterative two-grid methods. We also present some numerical experiments to confirm the efficiency of the proposed algorithm.

Mathematics subject classification: 65N30, 65M12, 35J60.
Key words: Iterative two-grid method, Convergence, Strongly nonlinear elliptic problems.

1. Introduction

Two-grid methods are first proposed for nonselfadjoint problems and indefinite elliptic problems [6, 10]. Then, two-grid methods are extended to solve semilinear elliptic problems [7], quasi-linear and nonlinear elliptic problems [8, 9], respectively. Especially, for nonlinear elliptic problems, the basic idea of two-grid methods is to first obtain a rough solution by solving the original problem in a coarse mesh with mesh size H, and then correct the rough solution by solving a symmetric positive definite (SPD) system in a fine mesh with mesh size h. Noticing the coarse mesh could be extremely coarse in contrast to the fine mesh, it is not difficult to solve an original problem in coarse mesh. Therefore, two-grid methods reduce the computational complexity of solving the original problem to solving a SPD problem and dramatically improve the computational speed. Recently, Bi *et al.* [1] presented a two-grid algorithm to solve the strongly nonlinear elliptic problems and provided a posteriori error estimator for the two-grid methods. It is noted that the literature mentioned above is all about non-iterative two-grid methods.

As is well-known, the mesh size H of coarse mesh and h of fine mesh should satisfy a certain relationship for the optimal convergence order in non-iterative two-grid methods. The iterative two-grid methods have the advantage over the non-iterative two-grid methods in that, the distance between the mesh sizes H and h can be enlarged by increasing the iteration counts with the same accuracy. However, there is only a small amount of literature on iterative

* Received April 16, 2023 / Revised version received July 31, 2023 / Accepted December 5, 2023 / Published online April 13, 2024 /
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two-grid methods of conforming finite element discretization for elliptic problems. Xu [9] first proposed and analyzed an iterative two-grid method for non-symmetric positive definite elliptic problems. Zhang *et al.* [11] designed some iterative two-grid algorithms for semilinear elliptic problems and provided the corresponding convergence analysis. To our knowledge, there is not any published literature on the iterative two-grid algorithm of conforming finite element discretization for strongly nonlinear elliptic boundary value problems.

In this paper, an iterative two-grid algorithm for solving strongly nonlinear elliptic problems is studied. The discrete system of strongly nonlinear elliptic problems is presented at first. And then, an iterative two-grid algorithm is proposed for the discrete system, which is obtained by applying a non-iterative two-grid algorithm of [8] in a successive fashion. Finally, a challenging convergence analysis of the proposed algorithm is provided. Despite the fact that our algorithm is simply obtained by [8], the convergence analysis of the non-iterative two-grid algorithm could not be directly applied to the iterative two-grid algorithm. Here we complete this challenging convergence analysis by mathematical induction which can also be used in solving semilinear elliptic problems by iterative two-grid algorithms in [11]. However, we must emphasize that the convergence analysis of our algorithm is significantly different from the one of [11]. Compared with the current work [11], our convergence analysis is far more difficult and complex, and specific challenges could be reflected in:

(1) the higher order derivative component of our model problem is still nonlinear,

(2) the coupled error estimates cause formidable obstacles for the convergence analysis (see the proof of Lemma 4.5).

To avoid the repeated use of generic but unspecified constants, $x \lesssim y$ is used to denote $x \leq Cy$, where C are some positive constants which do not depend on the mesh size. Furthermore the constants C may denote different values under different circumstances. For some specific constants, we use the constant C with some subscript to denote.

The rest of the paper is organized as follows. In Section 2, the discrete scheme of strongly nonlinear elliptic problems, as well as the corresponding well-posedness and a priori error estimates, are introduced. In Section 3, an iterative two-grid algorithms is proposed. And then some preliminaries and the convergence analysis of the proposed algorithms are provided in Section 4. Finally, some numerical experiments are presented to verify the efficiency of the proposed algorithm in Section 5.

2. Model Problems and Discrete Systems

In this section, we present the continuous and discrete variational problems of strongly nonlinear elliptic problems, and provide the corresponding well-posedness and a priori error estimates.

Given a bounded convex polygonal domain $\Omega \subset \mathbb{R}^2$ with the boundary $\partial\Omega$. We denote $W^{m,p}(\Omega)$ as the standard Sobolev space with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$, where the integers $m \geq 0$ and $p \geq 1$. For convenience, we also denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2,\Omega}$ and $H_0^1(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$.

We consider the following strongly nonlinear elliptic problem:

$$\begin{cases} -\nabla \cdot \boldsymbol{a}(\boldsymbol{x}, u, \nabla u) + f(\boldsymbol{x}, u, \nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$