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ERROR ANALYSIS OF FRACTIONAL COLLOCATION METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH NONCOMPACT OPERATORS*

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Abstract

This paper is concerned with the numerical solution of Volterra integro-differential equations with noncompact operators. The focus is on the problems with weakly singular solutions. To handle the initial weak singularity of the solution, a fractional collocation method is applied. A rigorous *hp*-version error analysis of the numerical method under a weighted H^1-norm is carried out. The result shows that the method can achieve high order convergence for such equations. Numerical experiments are also presented to confirm the effectiveness of the proposed method.

Mathematics subject classification: L05, L60.
Key words: Volterra integro-differential equation, Noncompact operator, Nonsmooth solution, Collocation method, Fractional polynomial, *hp*-version error analysis.

1. Introduction

Volterra integro-differential equations (VIDEs) arise in mathematical models of many different research fields, such as population models [28], viscoelastic phenomena [19], capillarity theory [4]. In this paper, we consider the VIDEs of the form

$$t^\gamma u'(t) = a(t)u(t) + g(t) + \int_0^t (t-s)^{-\mu} s^{\mu+\gamma-1} K(t,s)u(s)ds, \quad t \in I := [0,T] \tag{1.1}$$

with the initial condition $u(0) = u_0$, where $0 \le \mu < 1, \gamma > 0, \mu+\gamma \ge 1$, and $a(t), g(t)$ and $K(t,s)$ are given smooth functions. The Eq. (1.1) can be equivalently written as the following

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cordial Volterra integro-differential equation (CVIDE):

$$u'(t) = a_\gamma(t)u(t) + g_\gamma(t) + (K_{\mu,\gamma}u)(t), \quad t \in I := [0,T], \tag{1.2}$$

where

$$a_\gamma(t) = t^{-\gamma}a(t), \quad g_\gamma(t) = t^{-\gamma}g(t),$$

and

$$(K_{\mu,\gamma}u)(t) = t^{-\gamma}\int_0^t (t-s)^{-\mu}s^{\mu+\gamma-1}K(t,s)u(s)ds.$$

The operator $K_{\mu,\gamma}$ can be viewed as a cordial operator defined in [29, 30], where the author pointed out that such an operator is noncompact if $K(0,0) \neq 0$. One can also see that when $\gamma = 0$, the Eq. (1.2) reduces to a second kind VIDE.

There have been many researches on the numerical solution for several different classes of Volterra integral or integro-differential equations, such as collocation methods [3, 9, 11, 13, 26, 36, 38, 39], discontinuous Galerkin methods [17], block boundary value methods [40], spectral collocation methods [6, 10, 35, 37], spectral Galerkin methods [8, 27], *hp*-version collocation or Galerkin methods [15, 16, 32, 33]. All the above studies focus on the second kind Volterra-type equations.

For third kind VIDEs with form (1.1) or (1.2), however, there are only few works. One can see the study for the case that $\mu = 0$ in [12] and for the case that $K_{\mu,\gamma}$ is compact in [22]. For CVIDEs with noncompact cordial operators, continuous piecewise-polynomial collocation methods were considered in [25], where the convergence and superconvergence of the method were analysed. Based on smooth transformation, the Legendre spectral collocation method was employed in [14]. A related topic is the numerical solution of Volterra integral equations (VIEs) with the integral operator $K_{\mu,\gamma}$, which are also referred as third-kind VIEs [5, 18, 20]. For the latter, collocation methods [1, 31], multistep collocation methods [21] and Legendre Galerkin spectral methods [2] have received attention. Recently, an *hp*-version method, which can provide a flexible choice of locally varying time steps and approximation orders, was developed for solving third-kind VIEs in [34]. To the best of our knowledge, *hp*-version methods have not been considered for the VIDEs of the form (1.2), although such kind of methods has been widely studied for solving second kind VIDEs and VIEs.

In this paper, we apply a fractional collocation method to the Eq. (1.2) with noncompact cordial operator and nonsmooth solution. The method is based on piecewise fractional polynomial collocation with fractional exponent λ $(0 < \lambda \leq 1)$ which is a user-chosen parameter. Different from the classical polynomial collocation method, the approximation spaces for fractional collocation method are constructed using fractional polynomials of the form $\sum_{k=0}^{N} c_k t^{k\lambda}$ instead of standard polynomials. The motivation for using such spaces is that when the solution exhibits weak singularity at the initial point $t = 0$, the present fractional approximation spaces with a suitable λ can match well with this kind of singularity appearing in the solutions. We mention that such spaces have previously been used for the numerical solution of second kind weakly singular VIDEs, see for example [8, 9, 16].

For the proposed fractional collocation method, an *hp*-error estimate is established under a weighted H^1-norm. The error bound explicitly depends on the local time steps, the local approximation orders, and the regularity of $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$. Notice that for typical weakly singular solutions, $u(t^{1/\lambda})$ and $u'(t^{1/\lambda})$ can have a better regularity than the original solution $u(t)$ for suitable λ. This means that fractional collocation can achieve high order of convergence even for weakly singular solutions.