THE COUPLING OF BOUNDARY INTERAL AND FINITE ELEMENT METHODS FOR THE NAVIER–STOKES EQUATIONS IN AN EXTERIOR DOMAIN

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Abstract

In this paper, a technique of coupling variational formulation of FEM and BIE (boundary integral equation) is used to deal with stationary Navier-Stokes equations in an unbounded domain. We discuss well-posedness for the coupling variational problem, the regularization method and FEM-BEM approximation. Finally, operator splitting and optimal control techniques are used to treat the difficulty of nonlinearity and constraints in computer implementation.

1. Introduction

The coupling of FEM and BIE has recently been recognized as a powerful tool for solving a certain class of physical problems with an unbounded domain for which the traditional numerical analysis techniques are unsuitable.

Following basically A. Sequira etal. [1], [2] concerning Stokes case, the major aim of the present work is to develop this method for N–S equations in an unbounded domain. Essentially, the coupling method involves the choice of an artificial smooth boundary separating the unbounded domain into two regions; an integral equation over this interface, representing the solution in the exterior domain in terms of a single layer potential, is incorporated into a variational formulation in the primitive variable velocity-pressure for the interior region. This allows discretization along the artificial boundary together with a typical discretization by the FEM.

2. Statement of the Problem

The stationary N–S equations with an exterior domain are given as

$$\begin{cases} (u^j\nabla_j)u^i = \nabla_j\sigma^{ij} + f^i, \quad i = 1,2,\ldots,n, n = 2 \text{ or } 3, \quad \text{in } \Omega', \\ \text{div } u = 0 \quad \text{in } \Omega', \\ u|_\Gamma = u_0, \quad u \to u_\infty, \quad x \to +\infty, \quad \int_\Gamma u_0 ds = 0, \end{cases} \tag{2.1}$$

where Ω' is the exterior of a simply-connected bounded open set Ω in R^n with smooth boundary Γ, u the velocity of fluids, $p = p/\rho$ the pressure, f the external forces and $\lambda = \text{Re}^{-1}, \text{Re} = u_\infty L/\nu$ Reynolds number, σ^{ij}, σ_{ij} stress tensors, e^{ij}, e_{ij} strain rate tensors, ∇_i, ∇^i covariant and contravariant derivatives respectively, g_{ij}, g^{ij} metric tensors,

$$\begin{cases} \sigma_{ij}(u,p) = -pg_{ij} + 2\mu e_{ij}(u), \quad e_{ij}(u) = (\nabla_i u_j + \nabla_j u_i)/2, \\ \sigma^{ij}(u,p) = g^{ik}g^{jm}\sigma_{km}, \quad e^{ij}(u) = g^{ik}g^{jm}e_{km}. \end{cases}$$

We only consider the homogeneous boundary condition in the sequel, but all the results stated here still hold if the trace of u on Γ is any given sufficiently smooth function that admits a solenoidal extension (div $u = 0$) in Ω'.

Let $\Omega' = \Omega_1 \cup \Omega_2$ be a decomposition of the domain such that Ω_1 and Ω_2 are open subsets of Ω'. Γ_2 is their common smooth boundary with a unit normal exterior to $\Omega_2; \Omega_1$ is bounded and $\text{supp}(f) \subset\subset \Omega_1$.

It is well known [8] that there exists at least one solution for problem (2.1). Generally speaking, velosity or its gradient in subdomain Ω_2 is small in the amplitude compared with that in subdomain Ω_1. Therefore, the inertia term $u\nabla u$ in Ω_2 can be neglected, and problem (2.1) can be replaced by the following

$$\begin{cases} (u^j\nabla_j)u^i - \nabla_j\sigma^{ij}(u,p) = f^i, & \text{in } \Omega_1, \\ \text{div } u = 0, & \text{in} \Omega_1, \end{cases} \tag{2.2}$$

$$\begin{cases} \nabla_j\sigma^{ij}(u,p) = 0, & \text{in } \Omega_2, \\ \text{div } u = 0, & \text{in } \Omega_2, \end{cases} \tag{2.3}$$

$$U|_\Gamma = 0, \qquad u|_{\Gamma_2}^+ = u|_{\Gamma_2}^-, \tag{2.4}$$

where the last conditions represent the appropriate assembling of the two separate problems in Ω_1 and Ω_2.

3. Variational Formulation for the Continuous Coupling Problem

In order to reduce the problem in Ω_2 into an integral equation over the boundary, the fundamental solution $\{U^{ij}, P^i\}$ of the stationary Stokes equation with the concentrated force will be employed and can be expressed in arbitrary curvilinear coordinates as [8]