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ANALYSIS OF FLOW DIRECTED ITERATIONS*

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Abstract

Iterative Methods are studied for the solution of difference schemes for convection dominated flow problems.

§1. Introduction

The numerical solution of convection diffusion flow problems is of considerable difficulty. It is well known that when the diffusion coefficient is small (the case of convection dominated flow), it is hard to obtain accurate difference schemes; the presence of rapid transitions, or boundary layers, in the solution severely degrades the accuracy of the approximate solution. One may ask whether difficulties are also encountered in the numerical solution of the difference equations when the diffusion coefficient is small. In this paper we consider some difference approximations to the convection diffusion equation and we treat block Gauss Seidel iterations for the solution of these problems. We study the effect of the partitioning and ordering of the unknowns on the convergence of the Gauss-Seidel iterations. We find that, for convection dominated flow problems, the spectral radius of the iteration matrix is not an appropriate indicator of the convergence properties of the method; it is better to use a norm of the iteration matrix. Also, we find that sweeping the mesh in the direction of the underlying flow enhances the convergence of the Gauss Seidel iterations. In one dimension it is not hard to devise an algorithm to implement this idea. In two dimensions, we give a general procedure to automate the partitioning and ordering phase of the solution process. The general procedure is described using the graph of the matrix.

§2 contains some remarks about the Gauss Seidel method. In §3 we discuss the one dimensional problem. For the basic upwind difference scheme on a uniform

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mesh we find that if the unknowns are swept in the direction of flow, the norm of the iteration matrix satisfies the inequality $\|N\| \leq c\varepsilon/h^s$ where $s = 2$ or 3. Thus while the iterations in any event converge, if $\varepsilon << h$, the iterations converge very quickly. Since, with $\varepsilon << h$, the difference equations are basically solving the reduced problem, which is a first order equation, it may not be too surprising that the convergence is fast. However we find that this fast convergence also holds for difference schemes that are especially adapted to the solution of the convection diffusion problem, such as the exponential scheme of Southwell, Allen, and A.M. Il'in and a difference scheme of Ervin and Layton that has been found to provide good resolution of interior layers. We have performed numerical tests for these schemes, and also for discretizations with a refined mesh that is designed to capture the boundary layer. In the latter case, flow directed iterations do not perform as well, but they are better than iterating with other orderings of the unknowns. §4 deals with the basic upwind scheme in two dimensions. Here we find the same inequality for $\|N\|$ if flow directed iterations are used. The difficulty lies in ordering the nuknowns to implement flow directed iterations. We find an algorithm that solves this problem. Tests of this algorithm, and of other ordering procedures for two dimensional problems, are not given in this paper.

The conclusion of this study is that flow directed iterations may be a good way to solve the discrete equations arising from modelling convection dominated flow. If the boundary or interior layers are captured by refining the mesh, the convergence properties of the method are not so favorable. Further work on flow directed iterations must be done to deal with refined meshes, and to develop the method for nonlinear problems and for the Navier Stokes equations, which contain a continuity equation as well as equations of convection diffusion type.

In [8], Strikwerda has considered SOR methods for the iterative solution of convection diffusion problems. We describe his approach in §4. Some recent work of Goldstein on the use of preconditioned conjugate gradients for solving convection diffusion equations is given in [6].

§2. Some properties of the Gauss Seidel Method

We recall the Gauss Seidel method for solving a linear system $Au = f$. Let us write $A = D - L - U$, where, typically, D is a nonsingular diagonal or block diagonal matrix and, for some permutation matrix, P, PLP^{-1} and PUP^{-1} are respectively lower and upper block triangular. The Gauss Seidel iterations may be written $Du^{k+1} = Lu^{k+1} + Uu^k + f$ or, solving for $u^{k+1}, u^{k+1} = Nu^k + (D-L)^{-1}f$, where $N = (I - D^{-1}L)^{-1}D^{-1}U$ is the Gauss Seidel iteration matrix. Since $D^{-1}L$ is typically block triangular, $D^{-1}L$ is usually nilpotent. We say that $D^{-1}L$ is nilpotent of index m if $(D^{-1}L)^m = 0$, and if m is the smallest index for which this holds. We shall frequently use the following simple lemma to estimate $\|N\|$.