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ON THE SUFFICIENT CONDITIONS FOR THE SOLUBILITY OF ALGEBRAIC INVERSE EIGENVALUE PROBLEMS *1)

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Abstract

With the help of Brouwer's fixed point theorem and the relations of the eigenvalues and diagonal elements of a Hermitian matrix, we give some new sufficient conditions for the solubility of algebraic inverse eigenvalue problems.

§1. Introduction

We are interested in solving the following inverse eigenvalue problems:

Problem A (Additive inverse eigenvalue problem). Given an $n \times n$ Hermitian matrix $A = [a_{ij}]$, and n real numbers $\lambda_1, \cdots, \lambda_n$, find a real $n \times n$ diagonal matrix $D = \text{diag}\,(c_1, \cdots, c_n)$ such that the matrix $A + D$ has eigenvalues $\lambda_1, \cdots, \lambda_n$.

Problem M (Multiplicative inverse eigenvalue problem). Given an $n \times n$ positive Hermitian matrix $A = [a_{ij}]$, and n positive real numbers $\lambda_1, \cdots, \lambda_n$, find an $n \times n$ positive definite diagonal matrix $D = (c_1, \cdots, c_n)$ such that the matrix DA has eigenvalues $\lambda_1, \cdots, \lambda_n$.

Problem G (General inverse eigenvalue problem). Given $n+1$ complex $n \times n$ Hermitian matrix $A = [a_{ij}]$, $A_k = [a_{ij}^{(k)}]$, $k = 1, \cdots, n$, and n real numbers $\lambda_1, \cdots, \lambda_n$, find n real numbers $c_1, \cdots, c_n$, such that the matrix $A(c) = A + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, \cdots, \lambda_n$.

A number of sufficient conditions for those problems to have a solution have been discovered by many authors (see [1]-[3], [5]-[7]) . In the present paper we shall give some new sufficient conditions for those three problems to have solutions. These results are not contained in the presently known results and are better than the known results in some aspects.

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Notation and Definitions. Throughout this paper we use the following notation. $\mathbb{R}^n$ is the set of all n-dimensional real column vectors and ϕ is an empty set. The superscripts T and H are for transpose and conjugate transpose, respectively. $\rho(A)$ and $\mathrm{tr}(A)$ denote the spectral radius and the trace of a matrix A, respectively. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of a Hermitian matrix A, respectively. The norm $\| \quad \|_\infty$ stands for max-norm of a vector and maximum row sum matrix norm.

For arbitrary $n \times n$ matrices $B = [b_{ij}]$ and vectors $b = (b_1, \cdots, b_n)^T \in \mathbb{R}^n$, let

$$d(b) = \min_{i \neq j} |b_i - b_j|$$

and

$$B^{(0)} = B - \mathrm{diag}\,(b_{ii}).$$

Without loss of generality we can assume that $a_{jj}^{(k)} = \delta_{kj}$ for $k, j = 1, 2, \cdots, n$ in Problem G (see [1]), that $a_{ii} = 1$ for $i = 1, 2, \cdots, n$ in Problem M, and that $a_{ii} = 0$ for $i = 1, 2, \cdots, n$ in Problem A.

§2. Main Results

Theorem 1. *Suppose that* $\lambda_n > \cdots > \lambda_1 > 0, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ *in Problem M. If*

$$d(\lambda) \geq \lambda_n(\lambda_{\max}(A^{(0)}) - \lambda_{\min}(A^{(0)})), \tag{2.1}$$

then there exist n real numbers $c_1, \cdots, c_n$ such that the matrix DA has eigenvalues $\lambda_1, \cdots, \lambda_n$, where $D = \mathrm{diag}\,(c_1, \cdots, c_n)$.

Theorem 2. *Suppose that* $\lambda_n > \cdots > \lambda_1, \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ *in Problem G. If*

$$d(\lambda) \geq \lambda_n^* - \lambda_1^* + \lambda_0 \sum_{k=1}^{n} (\lambda_{\max}(A_k^{(0)}) - \lambda_{\min}(A_k^{(0)})), \tag{2.2}$$

then there exist n real numbers $c_1, \cdots, c_n$ such that the matrix $A + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, \cdots, \lambda_n$, where

$$\lambda_1^* = \lambda_{\min}\Big(A^{(0)} - \sum_{k=1}^{n} a_{kk} A_k^{(0)}\Big), \quad \lambda_n^* = \lambda_{\max}\Big(A^{(0)} - \sum_{k=1}^{n} a_{kk} A_k^{(0)}\Big),$$

$$\lambda_0 = \max\{|\lambda_1|, |\lambda_n|\}.$$

Applying Theorem 2 to the additive inverse eigenvalue problem, we get the following corollary.