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ALGORITHMS FOR INVERSE EIGENVALUE PROBLEMS*1)

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Abstract

Two new algorithms based on QR decompositions (QRDs) (with column pivoting) are proposed for solving inverse eigenvalue problems, and under some non-singularity assumptions they are both locally quadratically convergent.

Several numerical tests are presented to illustrate their convergence behavior.

§1. Introduction

Inverse eigenvalue problems arise often in applied mathematics (see [1], §1), and they are treated by many mathematicians. Let A be a fixed $n \times n$ (complex) valued matrix. The most common inverse eigenvalue problems are the following two problems proposed by Downing and Householder [11]:

(i) Find a diagonal complex valued matrix D such that the spectrum of $A + D$ is a given set $\lambda^* = (\lambda_1^*, \cdots, \lambda_n^*)$.

(ii) Find a diagonal complex valued matrix D such that the spectrum of AD is a given set $\lambda^* = (\lambda_1^*, \cdots, \lambda_n^*)$.

The first problem is called *the inverse additive eigenvalue problem*, and the second one *the multiplicative eigenvalue problem*. Often in practical applications, A, D and λ^* in the above two problems are real.

Notation. We shall use $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) for the m by n complex (real) matrix set, $\mathbb{C}^m = \mathbb{C}^{m\times 1}$ ($\mathbb{R}^m = \mathbb{R}^{m\times 1}$), $\mathbb{C} = \mathbb{C}^1$ ($\mathbb{R} = \mathbb{R}^1$); $\mathcal{U}_n \subset \mathbb{C}^{n\times n}$ denotes the n by n unitary matrix set. $I^{(n)}$ is the n by n unit matrix, $e_j^{(n)}$ the jth column of $I^{(n)}$ and $I_j^{(n)} \equiv (e_1^{(n)}, \cdots, e_j^{(n)})$. When no confusion arises, these superscripts (n) are usually omitted. A^H, A^T denote the conjugate transpose and transpose respectively, and $\|A\|_2$ the spectral norm of A.

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For convenience, in this paper we generalize the statement of the above two inverse problems as

Problem G. Let $A(c) \in \mathbb{C}^{n\times n}$ be a differentiable matrix-valued function of $c \in \mathbb{C}^n$. Find a point $c^* \in \mathbb{C}^n$ such that the spectrum of matrix $A(c^*)$ is a given set $\lambda^* = (\lambda_1^*, \cdots, \lambda_n^*)$.

Here, the differentiability of $A(c) \in \mathbb{C}^{n\times n}$ with respect to c means, for any $c^{(0)} \in \mathbb{C}^n$, we have

$$A(c) = A(c^{(0)}) + \sum_{i=1}^{n} \frac{\partial}{\partial c_i} A(c)\Big|_{c=c^{(0)}} (c_i - c_i^{(0)}) + o(\|c - c^{(0)}\|_2), \tag{1.1}$$

where

$$c = (c_1, \cdots, c_n)^T, \quad c^{(0)} = (c_1^{(0)}, \cdots, c_n^{(0)})^T, \tag{1.2}$$

$$\frac{\partial}{\partial c_i} A(c) = \left(\frac{\partial}{\partial c_i} a_{kj}(c) \right) \in \mathbb{C}^{n\times n} \quad \text{for} \quad A(c) = (a_{kj}(c))$$

and

$$\|c - c^{(0)}\|_2 = \left(\sum_{i=1}^{n} |c_i - c_i^{(0)}|^2 \right)^{\frac{1}{2}}. \tag{1.3}$$

As to the solvability, and some numerical methods of inverse additive, multiplicative eigenvalue problems, we refer the readers to, e.g. [7], [1]-[3] and other related references therein. The aim of this paper is to propose two new methods to solve Problem G and to analyze their convergence behavior under appropriate hypotheses. Throughout this paper we assume the given set λ^* satisfies

$$\lambda_i^* \neq \lambda_j^* \quad \text{for} \quad i \neq j, \tag{1.4}$$

and Problem G itself is solvable.

The rest of this paper is organized as follows: In §2 we cite some necessary differentiability theorems proved in [12]. In §3 we first discuss some formulations of numerical methods. Then we give our algorithms and their convergence analysis. Finally in §4 we present several numerical tests to illustrate their behavior.

§2. QR Decomposition (QRD) and Differentiability

Let $A(c) \in \mathbb{C}^{n\times n}$. The QRDs with column pivoting (see [9, pp.163-167]) of $A(c)$ can be read as

$$A(c)\pi(c) = Q(c)R(c), Q(c) \in \mathcal{U}_n, \tag{2.1}$$

where $\pi(c) \in \mathbb{C}^{n\times n}$ is a permutation matrix and $R(c) \in \mathbb{C}^{n\times n}$ an upper triangular matrix. The following theorem was obtained in [12], and it is the basis of this paper.