Journal of Computational Mathematics, Vol.10, No.3, 1992, 209–223.

A NEW APPROACH TO STABILITY OF FINITE ELEMENTS UNDER DIVERGENCE CONSTRAINTS*[1)]

Zhou Tian-xiao
(*Computing Institute, Chinese Areonautical Establishment, Xi'an, China*)
Feng Min-fu Xiong Hua-xin
(*Department of Mathematics, Sichuan University, Chengdu, China*)

Abstract

A new stability inequality for velocity-pressure F.E. approximations of Stokes (or Navier-Stokes) problems is presented in this paper. It is proved that the inequality holds if the so-called patch test of rank non-deficiency is passed. As a use of the new criterion, the stability of various new and old combinations of velocity interpolations with pressure interpolations is discussed.

§1. Introduction

For finite element analysis of incompressible flow of viscous fluids, it is important that the stability inequality

$$\sup_{u\in U_h(\Omega)} \frac{\int_\Omega \operatorname{div} u \cdot p d\Omega}{\|u\|_{1,\Omega}} \geq C\|p\|_{L^2_p(\Omega)}, \quad p \in V_h(\Omega) \tag{1.1}$$

hold for F.E. velocity-pressure space $U_h(\Omega) \times V_h(\Omega) \subset (H^1_0(\Omega))^n \times L^2_0(\Omega)$.

Up to now, some efforts have been made for the construction of velocity-pressure finite element spaces and their stability analysis (see [3], [6–9], [11], [15]). In particular, the following macroelement condition was presented in [7, 9]:

H) (1.1) holds for a regular partition J_h under the condition that all of the macroelements M, i.e. the union of one or more neighboring elements, forms a new subdivision $\mathcal{M}_M$ of the domain Ω, and for each $h > 0$ and $\Phi^h \in V_h(\Omega), \exists\ u^h_M \in U_h(\Omega)$ with $u^h_M|_{\Omega\setminus M} = 0$ such that

$$\int_M \hat{\Phi} \operatorname{div} u^h_M d\Omega \geq \|\hat{\Phi}_M\|^2_{L^2(\Omega)}, \quad |u^h_M|_{1,M} \leq C\|\hat{\Phi}_M\|_{L^2(M)}$$

and

$$\sup_{v\in U_h(\Omega)} \frac{\int_\Omega \operatorname{div} v\bar{\Phi}}{\|v\|_{1,\Omega}} \geq C\|\bar{\Phi}\|_{L^2(\Omega)}$$

* Received November 19, 1988.
1) Project supported by Chinese Aeronautical Science Foundation.

where $\bar{\Phi}_M = \frac{1}{|M|}\int_M \Phi^h d\Omega, \forall M \in \mathcal{M}_M, \hat{\phi}_M = (\Phi^h - \bar{\Phi})|_M$, the constant C is independent of h, Φ^h and u_M^h.

This condition is local. Stenberg [9] pointed out that it can be used to determine the stability of various combinations. It is, however, not "primitive". It seems to have no more adaptation than the condition of patch type used widely in finite elements owing to the requirement that the partition be composed of macroelements.

As an improvement of the macroelement condition H), a new stability condition is presented in this paper, which is also local and has the same feature of rank non-deficiency condition used in finite element analysis of solid mechanics, and can reduce the judgment of the stability condition (1.1) on macroelements to the determination on element patches; therefore there is no restriction on the partition.

The main result in this paper is the following:

H)′ If for each $p \in V_h(\Omega)$ and each possible element patch $M, (\operatorname{div} v, p)_{(M)} = 0$ $\forall v \in U_h(M) \subset (H_0^1(M))^n$ implies $p|_M =$ constant (i.e. rank non-deficiency), then there exists a constant C independent of the mesh h of finite element such that

$$\sup_{u\in U_h(\Omega)} \frac{\int_\Omega \operatorname{div} u \cdot p d\Omega}{\|u\|_{1,h,\Omega}} \geq C\|p\|_{0,h,\Omega}, \quad p \in V_h \tag{1.2}$$

where norms $\|\cdot\|_{l,h,\Omega}$ and $\|\cdot\|_{0,h,\Omega}$, which will be defined in Section 2, are different from that in (1.1). Though (1.2) is not identical with (1.1), it can be proved that the inequality can play the same role as (1.1). In many cases, we can easily give the proof for (1.2), but not for (1.1).

On the basis of the new criterion, a new stable combination of piecewise linear velocity and piecewise constant pressure is constructed in Section 4. The same conclusions are extended to the three-dimensional case, and the stability of various combinations of velocity-pressure discussed in [6, 9, 15] will be checked one by one in a simple and unified manner by virtue of the new stability condition.

§2. New Stability Inequality

2.1. New stability inequality

Let Ω be a convex polygonal domain with the boundary Γ in $R^n (n = 2, 3)$. The stationary Stokes equation is to find $u = (u_1, u_2, \cdots, u_n)$ and p such that

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \end{cases} \tag{2.1.1}$$

where u is the velocity vector, p is the pressure, and f is the body force. The above problem is equivalent to the following variational problem: