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THE DIMENSIONS OF SPLINE SPACES AND THEIR SINGULARITY*

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Abstract

In this paper, the dimensions of spaces $S_k^\mu(\Delta_n)(k \geq 2^n\mu + 1)$ are obtained, where Δ_n is a general simplicial partition of a bounded region with piecewise linear boundary. It is also pointed that the singularity of spaces $S_k^\mu(\Delta_n)$ can not disappear when $n \geq 3$ no matter how large k is. At the same time, a necessary and sufficient condition that Morgen and Scott's structure is singular is obtained.

§1. Dimension

Let $D_n \in I\!R^n$ be a bounded region with piecewise linear boundary, Δ_n a simplicial partition of D_n, and $S_j^{(i)}(i = 0,1,\cdots,n; j = 1,2,\cdots,T_i)$ be all i-simplices of Δ_n. $R(S^{(i)}) = \cup\{S^{(n)} \in \Delta_n : S^{(i)} \subset S^{(n)}\}$ is called an i-incident region of $S^{(i)}$.

Definition 1. *Let $S^{(i)} \in \Delta_n$ and P_0 be an inner point of $S^{(i)}$. Then*

$$T(S^{(i)}) = M \cap R(S^{(i)})$$

is called a transversal surface of $S^{(i)}$, where $M = \{P - P_0 \in I\!R^n : (P - P_0, V - P_0) = 0, V \in S^{(i)}\}$.

When $n = 2$ and $k \geq 4\mu + 1$,

$$\dim S_k^\mu(\Delta_2) = \frac{1}{2}(k - 3\mu - 1)(k - 3\mu - 2)T_2 + \frac{1}{2}(\mu + 1)(2k - 7\mu - 2)T_1 + \sum_{i=1}^{T_0} \dim S_{2\mu}^\mu(R(S_i^{(0)})), \tag{1}$$

(see [1] and [2]) and when $n = 3$ and $k \geq 8\mu + 1$,

$$\begin{aligned}\dim S_k^\mu(\Delta_3) = {} & (C_{k-4\mu-1}^3 - 4C_\mu^3)T_3 + \frac{1}{2}(\mu+1)((k-5\mu-1)(k-4\mu-2)+2\mu)T_2 \\ & + \sum_{i=1}^{T_1}\Big[(k-6\mu-1)\dim S_{2\mu}^\mu(T(S_i^{(1)})) - \sum_{j=0}^{2\mu-1}\dim S_j^\mu(T(S_i^{(1)}))\Big] \\ & + \sum_{i=1}^{T_0}\dim S_{4\mu}^\mu(R(S_i^{(0)})),\end{aligned} \tag{2}$$

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(see [3]) where $T(S_i^{(1)})$ is a transversal surface of $S_i^{(0)}$, and

$$C_m^n = \begin{cases} \dfrac{m!}{n!(m-n)!}, & \text{when} \quad m \ge n, \\ 0, & \text{otherwise.} \end{cases}$$

For a general case, we have

Theorem 1. *When $k \ge 2^n\mu + 1$,*

$$\dim S_k^\mu(\Delta_n) = \sum_{i=1}^{T_0} \dim S_{2^{n-1}\mu}^\mu(R(S_i^{(0)}))$$

$$+ \sum_{i=1}^{T_1}[(k - 3\cdot 2^{n-2}\mu - 1)\dim S_{2^{n-2}\mu}^\mu(T(S_i^{(1)})) - \sum_{j=0}^{2^{n-2}\mu-1} \dim S_j^\mu(T(S_i^{(1)}))]$$

$$+ \sum_{d=2}^{n-1}\sum_{j=1}^{T_d}[M(2^{n-d-1}\mu, d)\dim S_{2^{n-d-1}\mu}^\mu(TS_i^{(d)})) - \sum_{j=0}^{2^{n-d-1}\mu-1} (M(j+1,d)$$

$$- M(j,d))\dim S_j^\mu(T(S_i^{(d)}))] + M(0,n)T_n,$$

where $T(S_i^{(d)})$ is a transversal surface of $S_i^{(d)}$, and

$$M(d,i) = C_{A(i,d)}^{d} - (d+1)N(0,d,i) - \sum_{j=1}^{d-2} C_{d+1}^{j+1}[M(j, 2^{n-j-1}\mu)N(j,d,2^{n-d-1}\mu)$$

$$- \sum_{l=1}^{2^{n-d-1}\mu} (M(j, 2^{n-j-1}\mu - l + 1) - M(j, 2^{n-j-1}\mu - l))\cdot L(j,d,i,l)],$$

$$A(i,d) = k - (d+1)2^{n-d}\mu + d_i - 1, \quad M(1,i) = C_{k-2^n\mu-1+i}^1,$$

$$M(2,i) = C_{k-3\cdot 2^{n-2}+2i-1}^2 - 3N(0,2,i), \quad N(m,d,i) = C_{B(i,d,m)}^{d-m},$$

$$B(i,d,m) = 2^{n-m-1}\mu - (d-m)\mu\cdot 2^{n-d} + (d-m-1)i,$$

$$L(j,d,i,l) = C_{B(i,d,j)-l}^{d-j}.$$

To prove Theorem 1, we give the folowing interpolation conditions:

i) Let $S^{(0)} \in \Delta_n$, $\{S_1^{(1)}, \cdots, S_m^{(1)}\} \subset \Delta_n$ be all 1-simplices with $S^{(0)}$ as a common end point, and τ_i be the unit vector of $S_i^{(1)}$, if $S_{i_1}^{(1)}, \cdots, S_{i_n}^{(1)}$ are the edges of a n-simplex in Δ_n, then the given conditions are

$$\left\{\frac{\partial^{m_1}}{\partial\tau_{i_1}^{m_1}}\cdots\frac{\partial^{m_n}}{\partial\tau_{i_n}^{m_n}} f(S_{i_1,\cdots,i_n}^{(0)})\right\}, \quad 0 \le m_1 + \cdots + m_n \le 2^{n-1}\mu,$$

and if $V_1 = V[S_{b_1}^{(1)}, S_{a_2}^{(1)}, \cdots, S_{a_n}^{(1)}]$ and $V_2 = V[S_{b_2}^{(1)}, S_{a_2}^{(1)}, \cdots, S_{a_n}^{(1)}$ have a common $(n-1)$-simplex surface, then

$$\frac{\partial^{m_1}}{\partial\tau_{b_2}^{m_1}}\frac{\partial^{m_2}}{\partial_{a_2}^{m_2}}\cdots\frac{\partial^{m_n}}{\partial\tau_{a_n}^{m_n}} f(S_{b_2,a_2,\cdots,a_n}^{(0)})$$

$$= \left(d_1\frac{\partial}{\partial\tau_{b_1}} + \sum_{j=2}^{n} d_j\frac{\partial}{\partial\tau_{a_j}}\right)^{m_1}\frac{\partial^{m_2}}{\partial\tau_{a_2}^{m_2}}\cdots\frac{\partial^{m_n}}{\partial\tau_{a_n}^{m_n}} f(S_{b_1,a_2,\cdots,a_n}^{(0)})$$