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EXTRAPOLATION OF NYSTRÖM SOLUTIONS OF BOUNDARY INTEGRAL EQUATIONS ON NON-SMOOTH DOMAINS*

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Abstract

The interior Dirichlet problem for Laplace's equation on a plane polygonal region Ω with boundary Γ may be reformulated as a second kind integral equation on Γ. This equation may be solved by the Nyström method using the composite trapezoidal rule. It is known that if the mesh has $O(n)$ points and is graded appropriately, then $O(1/n^2)$ convergence is obtained for the solution of the integral equation and the associated solution to the Dirichlet problem at any $\underset{\sim}{x} \in \Omega$. We present a simple extrapolation scheme which increases these rates of convergence to $O(1/n^4)$.

§1. Extrapolation on Non-Uniform Meshes

In this paper we examine a technique for extrapolating numerical solutions (obtained by the Nyström method) to an integral equation defined on the boundary of a polygonal planar domain. At the corners of the boundary the integral operator and the solution suffer from loss of regularity, and the mesh should be graded to compensate for this.

We shall show that, even for such non-uniform meshes, extrapolation of the numerical solutions is still possible. For simplicity we shall restrict ourselves to the case that the Nyström scheme is based on the trapezoidal rule (since this has some computational advantages), but the results obtained should generalise easily to other commonly used quadrature rules.

To motivate the extrapolation procedure, consider first the simple quadrature problem for a function v over $[0,1]$. Let Π_n be an arbitrary mesh $0 = x_0 < x_1 < \cdots < x_n = 1$, and let $h_i = x_i - x_{i-1}$. Let $T_n v$ denote the composite trapezoidal rule with respect to Π_n applied to v. Assume v has sufficient derivatives. Then the Euler-Maclaurin series gives

$$T_n v - \int_0^1 v = \frac{B_2}{2}\sum_{i=1}^{n} h_i^2 \int_{x_{i-1}}^{x_i} D^2 v + R_n v, \tag{1}$$

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where B_2 is the second Bernoulli number, and where

$$|R_n v| \le C \sum_{i=1}^{n} h_i^4 \int_{x_{i-1}}^{x_i} |D^4 v| \tag{2}$$

for some C independent of n and i. Now let Π_{2n} be the mesh obtained by dividing each of the intervals of Π_n exactly in half, and let $T_{2n}v$ be the corresponding composite trapezoidal rule applied to v. It is clear from (1) that

$$(4T_{2n}v - T_n v)/3 - \int_0^1 v = (4R_{2n}v - R_n v)/3.$$

Hence, under the conditions that $D^4 v$ is integrable and $h_i \le C(1/n)$ for each i, we have, from (2),

$$\left|(4T_{2n}v - T_n v)/3 - \int_0^1 v\right| = O(1/n^4). \tag{3}$$

However, under the same conditions, direct use of (1) yields only

$$\left|T_n v - \int_0^1 v\right| = O(1/n^2).$$

Thus, one step of Richardson extrapolation has doubled the rate of convergence of the trapezoidal approximations to the integral of v. The conditions on v and Π_n under which this result holds may be weakened: It is easy to see that (3) remains true provided

$$\sum_{i=1}^{n} h_i^4 \int_{x_{i-1}}^{x_i} |D^4 v| = O(1/n^4),$$

a criterion which naturally leads to the selection of a graded mesh with smaller subintervals where v varies most rapidly.

In §3 we shall show that the same principle holds for Nyström-trapezoidal solutions of boundary integral equations on polygonal domains. It turns out that, provided the mesh reflects accurately enough the (known) qualitative behaviour of appropriate higher derivatives of the unknown solution of the integral equation, extrapolation, analogous to that described above, can be performed.

There has been a long history of interest in extrapolation as a means of accelerating the convergence of numerical solutions of differential and integral equations. In recent years, considerable progress has been made on the extrapolation of finite element solutions of partial differential equations (see e.g., [2], [5], [6] and the references therein).

The use of extrapolation in the solution of integral equations was initiated by Baker[1], who obtained asymptotic error expansions for trapezoidal-Nyström solutions of second kind equations with smooth or Green's function kernels. Baker then used his expansions to prove convergence of a deferred correction technique. The results of Baker have recently been considerably extended by McLean, who obtained in [8] asymptotic error expansions for a wide range of Nyström, collocation, and Galerkin solutions to smooth equations.