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ON THE NUMERICAL METHOD OF FOLLOWING HOMOTOPY PATHS*[1)]

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Abstract

In this paper, we develop one kind of method, called self-adaptive method (SAM), to trace a continuous curve of a homotopy system for the solution of a nonlinear system of equations in finite steps. The existence of the continuous solution, the determination of safe initial points, and the test of regularity and stop criterion corresponding to this method are discussed. As a result the method can follow the curve efficiently. The numerical results show that our method is satisfactory.

§1. Introduction

The homotopy extension method, as a kind of algorithm for finding the solution of nonlinear systems

$$F(x)=0, \quad F:D\subset R^n\to R^n, \tag{1}$$

is very important. Like to the simplicial algorithm, this method will attract more attention because both of them are continuation techniques for finding the fixed points or zeros , and are related with Newton's method.

The principal idea of the homotopy extension method is to transform (1) into the following form (2) by homotopy mapping:

$$H(x(t))=0 \tag{2}$$

for arbitrary $t\in[0,1]$, and to follow the continuous curves of (2). In this respect, many results have been presented, of which one inmportant result is the local convergence theorem on "Newton following" given by Oterga and Rheinboldt [5]. But a series of problems on the existence of the solution, the partition on "Newton following", the computer implementability and the regularity of $\partial_x H(x(t),t)$ have not been solved yet, and a lot of difficulties are yet to be overcome for the numerical procedure of homotopy extension.

In this paper, we develop a kind of method to follow a continuous curve of the solutions in finite steps of the homotopy systems by the self-adaptive method. The numerical result shous that the algorithm can be implemented on computer.

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§2. Existence of the Solutions

Consider the following nonlinear systems:

$$F(x) = 0, F : D \subset R^n \to R^n. \tag{3}$$

The famous Newton-Kantorovich theorem shows the relations between local pionts and the solution. Our idea is to trace the homotopy path by using the Newton-Kantorovich theorem, so we need the following facts.

Definition 1. *Let $f : D \subset R^n \to R^1$ be a mapping, and $\rho \in R^1$ be positive real. Define the set*

$$\Gamma_{D,f}(\rho) = \{x | f(x) \leq \rho, \forall x \in D\}. \tag{4}$$

We say that $\Gamma_{D,f}$ is a level set of f on D and ρ.

Definition 2. *We say that D_r is an r-interior set of $D_0 \subset D$, if D_r satisfies*

$$D_r = \{x | \min \|y - x\| \geq r, x \in D_0, Y \in \partial D_0\} \tag{5}$$

where ∂D_0 is the boundary of D_0 , and r is positive.

Theorem 1. *Let $F : D \subset R^n \to R^n$ be C_1 smooth and F' satisfy*

$$\|F'(x) - F'(y)\| \leq L\|x - y\|, x, y \in D_0. \tag{6}$$

For $x \in D_0, F'(x)$ is invertible and satisfies

$$\|F'(x)^{-1}\| \leq \beta. \tag{7}$$

Then there exist solutions if and only if there is a positive ε_0 such that

$$\operatorname{int}(\Gamma_{D_0}) = \operatorname{int}\left(\Gamma_{D_0, \|F'(x)^{-1}F(x)\|}(\eta)\right) \neq \phi \tag{8}$$

where η satisfy

$$\eta \leq \min\left\{\frac{1}{2\beta L}, \varepsilon_0 - \frac{1}{2}\beta L \varepsilon_0^2\right\} \tag{9}$$

and ϕ is empty.

Proof. If ε_0 and η satisfy (8) and (9), then the interior of Γ_{D_0} is nonempty. Let $x^0 \in \operatorname{int}(\Gamma_{D_0})$. By Definition 2, we have $x^0 \in D_{\varepsilon_0} \subset D_0$ and x^0 satisfies

$$\|F'(x^0)^{-1}F(x^0)\| \leq \eta.$$

By (9),

$$\alpha = \beta L \eta \leq \frac{1}{2} \quad \text{and} \quad \overline{S}(x^0, \varepsilon_0) \subset D_0.$$

Let

$$t^* = (1 - \sqrt{1 - 2\alpha})\eta/\alpha.$$

Then we have

$$t^* - \frac{1}{2}\beta L t^* = \eta \leq \varepsilon_0 - \frac{1}{2}\beta L \varepsilon_0^2.$$

Thus

$$(\varepsilon_0 - t^*)(1 + \frac{1}{2}\beta L(t^* - \varepsilon_0)) \geq 0.$$

So we get $\varepsilon_0 \geq t^*$, and thus $\overline{S}(x^0, t^*) \subset D_0$. Then x^0 satisfies all the conditions of the Newton-Kantorovich theorem, which gurantees the existence of the solution.