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A NEW TYPE OF REDUCED DIMENSION PATH FOLLOWING METHODS*

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Abstract

To solve $F(x) = 0$ numerically, we first prove that there exists a tube-like neighborhood around the curve in R^n defined by the Newton homotopy in which $F(x)$ possesses some good properties. Then in this neighborhood, we set up an algorithm which is numerically stable and convergent. Since we can ensure that the iterative points are not far from the homotopy curve while computing, we need not apply the predictor-corrector which is often used in path following methods.

§1. Introduction

Suppose $F : R^n \to R^n$ is a smooth mapping. Let us consider solving $F(x) = 0$ globally. Recently, homotopy methods are used, that is, homotopy $H(t, x) = 0$ implicitly defines a path or a curve which leads to the root x^* of $F(x) = 0$. By following this path, we can finally reach x^*. But on computing, to follow the curve closely, we must use the predictor-corrector, which, of course, may bring us some trouble.

Here we take the advantadge of the Newton homotopy, and set up a new path following algorighm which (i) is numerically stable and (ii) does not use the correction technique. With it we can also judge whether or not we are going along the path we are following. Thus, the algorithm might make up for the deficiencies in current path following methods.

In this paper, we will use the following notations:

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2 = (x_1, x_2, ..., x_n)^T \in R^n, \quad \|A\| = \max_{\|x\|=1} \|Ax\|, A \in L(R^n, R^m),$$

$$DQ = (\partial Q_i/\partial x_j), \quad Q : R^n \to R^m, \quad B(x, \delta) = \{y; \|y - x\| < \delta\}, x \in R^n,$$

$$d(y, E) = \inf\{\|y - x\|; x \in E\}, y \in R^n, E \subset R^n.$$

If $F(x)$ is a mapping from R^n to R^n, we then denote its last $n-1$ exponents by $G(x)$:

$$G(x) = (F_2(x), F_3(x), ..., F_n(x))^T.$$

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§2. Basic Curve and Its Properties

Let $x_0 \in R^n$ be a given point. Consider the Newton homotopy

$$H(t,x) = F(x) - (1-t)F(x_0), \qquad 0 \le t \le 1 \tag{2.1}$$

chow etal. [1] discussed the conditions under which the connected exponents of one-dimensional manifolds determined by $H(t,x) = 0$ will be a diffeomorphism. [2] then proved the following theorem:

Theorem 2.1. *If $F(x): R^n \to R^n$ and $\bar{M} > 0$ satisfy*

(1) *F is a C^2 and proper mapping;*

(2) *the Lebesque measure of $\{x; \det DF(x) = 0\}$ is zero;*

(3) $\det DF(x) \ge 0$, *as* $\|F(x)\| \ge \bar{M}$.

Then for almost all $x_0 \in E_+(M) = \{x; \|F(x)\| \ge \bar{M}\}$, the projection $x(t)$: $[0,1] \to R^n$, of curve $(t, x(t))$ defined by the Newton homotopy $H(t,x) = 0$, is a diffeomorphism and connects x_0 with set $\{x; F(x) = 0\}$. Especially, $\{x; F(x) = 0\}$ is nonempty.

With the above condition, by Sard's theorem, we obtain

$$\text{rank } DH(t,x) = n, \quad \text{for } (t,x) \in \{(t,x); H(t,x) = 0\}. \tag{2.2}$$

In the following, we will always assume that $\det DF(x) \ne 0$ for $x \in \{x; F(x) = 0\}$ and $F(x)$, and x_0, and $H(t,x)$ satisfy the hypotheses in Theorem 2.1 and (2,2). Then, we have a smooth curve $x(t): [0,1] \to R^n$, denoted by $C(x_0)$.

We further assume its arc length $L < +\infty$ and parametrize by arc length s, i.e., $x(s): 0 \le s \le L, x(0) = x_0, x(L) = x^*$, instead of t.

For given $F(x_0)$, there exists an orthogonal matrix P, such that

$$PF(x_0) = (\|F(x_0)\|, 0, \cdots, 0)^T.$$

Obviously, solving $PF(x) = 0$ is equivalent to solving $F(x) = 0$, and $PF(x)$ and $F(x)$ have the same properties along $C(x_0)$. Later, we will consider solving $PF(x) = 0$ instead of $F(x) = 0$ and still denote $PF(x)$ by $F(x)$. Thus we have

$$F(x_0) = (\|F(x_0)\|, 0, \cdots, 0)^T. \tag{2.3}$$

From (2.2) and (2.3), we can easily get

Lemma 2.2. *If $F(x)$ and x_0 satisfy the conditions and* (2.3), *then*

$$\text{rank } DG(x) = n-1, \quad \textit{for} \quad x \in \overline{C(x_0)}. \tag{2.4}$$

Hence, there is a constant $\gamma > 0$, such that

$$\|DG(x)^+\| \le \gamma, \quad x \in \overline{C(x_0)} \tag{2.5}$$

where $DG(x)^+ = DG(x)^T(DG(x)DG(x)^T)^{-1}$.

By the definitions of $H(t,x)$ and $G(x)$, we can get

Lemma 2.3. *If $F(x)$ and x_0 are as above, then*

(1) $F_1(x) \ge 0, G(x) \equiv 0$, *for* $x \in \overline{C(x_0)}$;