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A CLASS OF THREE-LEVEL EXPLICIT DIFFERENCE SCHEMES*

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Abstract

A class of three-level six-point explicit schemes L_3 with two parameters s, p and accuracy $O(\tau h + h^2)$ for a dispersion equation $U_t = aU_{xxx}$ is established. The stability condition $|R| \le 1.35756176$ ($s = 3/2$, $p = 1.184153684$) for L_3 is better than $|R| < 1.1851$ in [1] and seems to be the best for schemes of the same type.

Any three-level explicit difference scheme for a dispersion equation $U_t = aU_{xxx}$ can be written in the form

$$U_{m+s}^{n+1} = \sum_{j=i}^{k} b_j U_{m+j}^{n} + \sum_{j} c_j U_{m+j}^{n-1} \tag{*}$$

(*) is referred to as an "N-point" scheme, where $N = k - i + 1$ ($k > i$). A class of six-point schemes L_3 containing two paremeters s and p is established in this paper. Their local truncation errors are $O(\tau h + h^2)$. The optimal stability condition obtained is $|R| \le 1.35756176$ ($R = a\tau/h^3$, $\tau = \Delta t$, $h = \Delta x$), which corresponds to $s = 3/2$, $p = 1.184153684$. This stability condition is an improvement on the result $|R| \le 1.1851$ in [1] and seems to be the best condition for six-point schemes of the same type at present.

The schemes given in this note are as follows:

$$L_3: \quad U_{m+s}^{n-d+1} - U_{m+s}^{n-d} + U_{m-s}^{n+d} - U_{m-s}^{n+d-1} = 2R\sum_{j=0}^{2} C_j \left(U_{m-j+1/2}^{n} - U_{m-j-1/2}^{n}\right) \tag{1}$$

where $a > 0$ if $d = 0$, $a < 0$ if $d = 1$, $s = 1/2, 3/2$; $C_0 = 2.5p - 3$, $C_1 = -1.25p + 1$, $C_2 = 0.25p$.

For $s = 1/2$ and $p = 1$, the schemes L_3 become H_3 in [1].

Now we analyse the stability of schemes L_3 by the Fourier method. For definiteness, put $s = 3/2$, $d = 0$ ($a > 0$). Let

$$U_m^n = \lambda^n e^{iqx_m}, \quad i^2 = -1,\ x_m = mh,\ q\text{-real number}. \tag{2}$$

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Substituting (2) into (1), we obtain the characteristic equation of schemes L_3 (see, [3]):

$$e^{iQ}\lambda^2 - 2F(Q)i\lambda - e^{-iQ} = 0, \quad Q = qh/2, \tag{3}$$

$$F(Q) = 2R\sum_{j=0}^{2} C_j \sin(2j+1)Q + \sin(3Q)$$

$$= Rf(y,p) + g(y), \quad y = \sin Q, \quad 0 \le Q \le \pi/2,$$

$$f(y,p) = 8y^3(py^2-1) = 8y^3(y-c)(y+c)/c^2, \quad p > 1, pc^2 = 1, \tag{4}$$

$$g(y) = 3y - 4y^3, \quad 0 \le y \le 1. \tag{5}$$

From equation (3) and [2,4], it follows that the stability condition of L_3 is $|Rf(y,p)+g(y)| < 1$ or

$$|R| < \sup_p \inf_{0<y\le 1} G(y,p), \tag{6}$$

$$G(y,p) = \begin{cases} -(1+g(y))/f(y,p), & 0 < y < c, \\ (1-g(y))/f(y,p), & 0 < y \le 1. \end{cases} \tag{7}$$

In order to find $\inf G(y,p)$ in the interval $0 < y \le 1$ for any fixed $p > 1$, the properities of $G(y,p)$ are discussed in the following.

1. In the case $0 < y < c$, we have

$$\partial G/\partial y = 8y^2(2y+1)W(y,p)/f(y,p)^2,$$

$$W(y,p) = py^2(-4y^2+2y+5) - 3, \tag{8}$$

$$W(0,p) = -3, \quad W(c,p) = 2(2c+1)(1-c) > 0,$$

$$\partial W/\partial y = py(16y+10)(1-y) > 0,$$

$$\partial G/\partial p = 8y^5(1+g(y))/f(y,p)^2 > 0. \tag{9}$$

From the above equalities, we see that there exists a unique zero point z of $W(y,p)$ or $\partial G/\partial y$, and z is also a unique minimum point of $G(y,p)$ for $0 < y < c$ because $G(0,p)$, $G(c,p) \to \infty$, and $G(y,p)$ is obviously a monotonically increasing function of p for any $y \in (0,c)$ (see, (9)). Thus, for arbitrary numbers p_1, p_2, c_1, c_2 satisfying $p_1 > p_2$ and $p_1c_1^2 = p_2c_2^2 = 1$, we have $c_1 < c_2$, and

$$\inf_{0<y<c_1} G(y,p_1) = G(z_1,p_1) > G(z_1,p_2) \ge \inf_{0<y<c_2} G(y,p_2) = G(z_2,p_2). \tag{10}$$

This verifies that $\inf G(y,p)$ $(0 < y < c)$ is a monotonically increasing function of $p > 1$.

2. In the case of $c < y \le 1$, we have

$$\partial G/\partial y = 8y^2 H(y,p)/f(y,p)^2,$$

$$H(y,p) = pL(y) - 6y + 3, \quad 1 < p < p_0, \ c_0 < y \le 1, \tag{11}$$

$$L(y) = -8y^5 + 12y^3 - 5y^2 = 4y^2(2y-1)(y_1-y)(y-y_2),$$

$$y_1 = (\sqrt{21}-1)/4, \quad y_2 = -(\sqrt{21}+1)/4,$$