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MINIMAX METHODS FOR OPEN-LOOP EQUILIBRA IN N-PERSON DIFFERENTIAL GAMES PART II: DUALITY AND PENALTY THEORY*1)

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Abstract

The equilibrium strategy for N-person differential games can be obtained from a min-max problem subject to differential constraints. The differential constraints are treated here by the duality and penalty methods.

We first formulate the duality theory. This involves the introduction of $N+1$ Lagrange multipliers: one for each player and one commonly shared by all players. The primal min-max problem thus results in a dual problem, which is a max-min problem with no differential constraints.

We develop the penalty theory by penalizing $N+1$ differential constraints. We give a convergence proof which generalizes a theorem due to B.T. Polyak.

§1. Introduction

In part I[5], we have presented a new minimax approach to N-person nonzero-sum differential games. We have also seen several advantages of using this approach.

The constraint equation contains N strategy variables $u_1, \cdots, n_N$ and one state variable x. Although $2N$ auxiliary variables $v_1, \cdots, v_N$ and $x^1, \cdots, x^N$ have been added in N supplementary differential equation constraints, they play the same roles as $u_1, \cdots, u_N$ and x, respectively. The functional $F(u, v)$ depends on $u_1, \cdots, u_N, v_1, \cdots,$ and v_N; $N+1$ state variables $x, x^1, \cdots, x^N$ and $N+1$ differential constraints are eliminated by integration. Therefore, from the mathematical programming point of view, the approach taken in Part I can be classified as primal. Computationally, this involves a rather large number of quadrature evaluations[3].

It is fair for us to say that most works in the literature on minimax problems are primal in nature in the sense that their constraints are handled in an implicit way.

On the other hand, looking back at optimal control problems, we understand that the use of different mathematical programming approaches of duality and penalty (cf.

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[12], [19], [4]) can lead to significant insights for solutions of those problems. These approaches also have the added advantage of being very amenable to numerical computations. One may wonder what can be done for N-person differential games. Here we are interested in developing some duality and penalty theory for minimax problems as well as numerical methods for N-person differential games. Indeed, this is the main motivation of our work.

By duality or penalty, differential constraints are handled explicitly. In the duality method Lagrange multipliers are introduced which eliminate the state constraints. In the penalty method, the system dynamics equations are penalized, which again results in an unconstrained problem. Both methods invole fewer quadrature calculations, and the variational matrix equations are sparse. Thus the computation is less costly and more efficient.

In §2, we first establish the duality theory under a general setting. For N-person games, we need to introduce $N+1$ Lagrange multipliers: one for each player, and one commonly shared by all players. Under the convexity-concavity assumption, we use the Hahn-Banach separation theorem to prove that the primal inf-sup problem leads to a dual sup-inf problem.

In §3, we present the fundamental penalty theorem. $N+1$ auxiliary state equations are penalized, with $N+1$ penalty parameters. The rate of convergence with respect to the penalty parameters is determined. Our work here extends and generalizes an earlier result of B.T. Polyak [16].

The applications of duality and penalty theory to finite element and their numerical examples are given in Part III[6].

§2. Duality Theory

As in Part I, we assume the following linear dynamics:

$$\begin{aligned} &\dot{x}(t) - A(t)x(t) - \sum_{i=1}^{N} B_i(t)u_i(t) - f(t) = 0 \quad \text{on } [0,T], \\ &x(0) = x_0 \in R^n. \end{aligned} \tag{2.1}$$

For notational convenience later on, we denote the system differential equation as

$$(DE) \equiv \dot{x}(t) - A(t)x(t) - \sum_{i=1}^{N} B_i(t)u_i(t) - f(t).$$

The matrix and vector functions $A(t), f(t), B_i(t), u_i(t), i = 1, \cdots, N$, satisfy the same conditions as in Part I.

Each player wants to minimize his cost

$$J_i(x,u) = J_i(x,u_1,\cdots,u_N), \quad i = 1,2,\cdots,N \tag{2.2}$$

which is continuous with respect to (x,u) in the $H_n^1 \times U$ norm. As before, we let

$$F(x,u;X,v) = F(x,u_1,\cdots,u_N;x^1,\cdots,x^N,v_1,\cdots,v_N)$$