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OPTIMUM MODIFIED SOR (MSOR) METHOD IN A SPECIAL CASE*

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Abstract

In this paper we study the MSOR method with fixed parameters, when applied to a linear system of equations $Ax = b$ (1), where A is consistently ordered and all the eigenvalues of the iteration matrix of the Jacobi method for (1) are purely imaginary. The optimum parameters and the optimum virtual spectral radius of the MSOR method are also obtained by an analysis similar to that of [5, pp. 277-281] for the real case. Finally, a comparison of the optimum MSOR method with the optimum SOR and AOR methods is presented, showing the superiority of the MSOR one.

§1. Introduction

To solve the linear system of equations:

$$Ax = b, \tag{1.1}$$

where $A \in R^{n,n}$, $b \in R^n$ and $\det(A) \neq 0$, we consider the modified successive overrelaxation (MSOR) method with fixed parameters (see e.g. [5, Chapter 8], [2]). We also assume that A has the form

$$A = \begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix} \tag{1.2}$$

where D_1 and D_2 are nonsingular diagonal matrices. If we partition x and b in (1.1) in accordance with the partitioning of A in (1.2), we can write system (1.1) in the form

$$\begin{bmatrix} D_1 & H \\ K & D_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{1.3}$$

The MSOR method is defined by

$$x^{(m+1)} = L_{\omega,\omega'} x^{(m)} + z_{\omega,\omega'}, \quad m = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$L_{\omega,\omega'} = \begin{bmatrix} (1-\omega)I_1 & \omega F \\ \omega'(1-\omega)G & \omega\omega' GF + (1-\omega')I_2 \end{bmatrix} \tag{1.5}$$

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and

$$z_{\omega,\omega'} = \begin{bmatrix} \omega c_1 \\ \omega\omega' G c_1 + \omega' c_2 \end{bmatrix}. \tag{1.6}$$

In (1.5) and (1.6),

$$F = -D_1^{-1}H,\ G = -D_2^{-1}K,\ c_1 = -D_1^{-1}b_1,\ c_2 = -D_2^{-1}b_2, \tag{1.7}$$

I_1 and I_2 are identity matrices of the same sizes as D_1 and D_2 respectively and $\omega, \omega'(\neq 0)$ are the real relaxation factors. If $\omega = \omega'$, the MSOR method reduces to the SOR method and we write

$$L_\omega \equiv L_{\omega,\omega}.$$

In the following we first find necessary and sufficient conditions for strong convergence and then determine the optimum parameters and the optimum virtual spectral radius of the MSOR method under the assumption that the eigenvalues of the iteration matrix B of the Jacobi method for system (1.3) are all purely imaginary. For this purpose we follow an analysis similar to that given in [5, pp. 277-281] for the case where all the eigenvalues of B are real. For other results in the real case see also [4].

§2. Convergence Analysis

According to Theorem 2.1 [5,p.273] for the matrix $L_{\omega,\omega'}$ the following are true: (i) If λ is an eigenvalue of $L_{\omega,\omega'}$, then there exists an eigenvalue ξ of the iteration matrix B of the Jacobi method for system (1.3) (note that

$$B = \begin{bmatrix} 0_1 & F \\ G & 0_2 \end{bmatrix}, \tag{2.1}$$

where 0_1, 0_2 are square null matrices of the same sizes as D_1 and D_2 respectively), such that

$$(\lambda + \omega - 1)(\lambda + \omega' - 1) = \omega\omega'\xi^2\lambda\ . \tag{2.2}$$

(ii) If ξ is a nonzero eigenvalue of B and if λ satisfies (2.2), then λ is an eigenvalue of $L_{\omega,\omega'}$. If $\xi = 0$ is an eigenvalue of B, then $\lambda = 1-\omega$ and/or $\lambda = 1-\omega'$ is an elgenvalue of ω, ω' .

We can write (2.2) as follows:

$$\lambda^2 - b\lambda + c = 0\ , \tag{2.3}$$

where

$$c = (\omega - 1)(\omega' - 1), \quad b = \omega\omega'\xi^2 - \omega - \omega' + 2 = 1 + c - \omega\omega'(1 - \xi^2)\ . \tag{2.4}$$

Since $b = b(\xi^2)$, following §6.1 [5, p.170] we can define the virtual spectral radius of $L_{\omega,\omega'}$ by

$$\bar{\rho}(L_{\omega,\omega'}) = \max_{\xi^2 \in \bar{S}_{B^2}} \psi(\omega, \omega', \xi^2)\ , \tag{2.5}$$