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A QUASI-PROJECTION ANALYSIS FOR ELASTIC WAVE PROPAGATION IN FLUID-SATURATED POROUS MEDIA*

Chen Zhang-xin
(*Department of Mathematics, Purdue University, USA*)

Abstract

This paper deals with the superconvergence phenomena for Galerkin approximations of solutions of Biot's dynamic equations describing elastic wave propagation in fluid-saturated porous media. An asymptotic expansion to high order of Galerkin solutions is used to derive these results.

§1. Introduction

An isotropic, elastic porous solid saturated by a compressible viscous fluid can be described by the system of partial differential equations [1], [6],

$$\mathcal{A}\frac{\partial^2 u}{\partial t^2} + \mathcal{C}\frac{\partial u}{\partial t} - L(u) = F(x,t), (x,t) \in \Omega \times [0,T], \tag{1.1}$$

where Ω is a bounded domain in $\boldsymbol{R}^2$, $u(x,t) = (u_1, u_2)$ is the displacement vector on Ω, $u_1 = (u_{11}, u_{12})$ and $u_2(x,t) = (x_{21}, u_{22})$ are the displacement of the solid and the average fluid displacement, respectively, and $F(x,t)$ is the force applied to the system. The differential operator $L(u)$ is defined by

$$L(u) = (\nabla \cdot \theta_1(u), \nabla \cdot \theta_2(u), \nabla s(u)),$$

where the vectors $\theta_i(u), i = 1,2$, and the scalar $s(u)$ are

$$\theta_i(u) = (\theta_{i1}, \theta_{i2}),\ i = 1,2; \quad s(u) = Q\nabla \cdot u_1 + R\nabla \cdot u_2,$$

and

$$\theta_{ij}(u) = \sigma_{ij}(u_1) + Q\delta_{ij}\nabla \cdot u_2, \quad i,j = 1,2.$$

Here δ_{ij} denotes the Kronecker symbol, and the stress tensors σ_{ij} and the strain tensors ϵ_{ij} for Ω are related by

$$\varepsilon_{ij}(u_1)\frac{1}{2}\Big(\frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i}\Big), \ 1 \le i,\ j \le 2,$$

$$\sigma_{ij}(u_1) = A\delta_{ij}\sum_{k=1}^{2}\varepsilon_{kk}(u_1) + 2N\varepsilon_{ij}(u_1), \ 1 \le i,\ j \le 2.$$

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$A = A(x), N = N(x), Q = Q(x)$, and $R = R(x)$ are the elastic coefficients for Ω. They will be assumed to satisfy the constraints

$$0 < N_* \le N(x) \le N^* < \infty,\ x \in \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$0 < A_* \le A(x) \le A^* < \infty,\ x \in \bar{\Omega},$$

$$0 < Q_* \le Q(x) \le Q^* < \infty,\ x \in \bar{\Omega},$$

$$0 < R_* \le R(x) \le R^* < \infty,\ x \in \bar{\Omega},$$

$$R(A+N) - Q^2 > 0,\ x \in \bar{\Omega}.$$

In (1.1), $\mathcal{A} \in \boldsymbol{R}^{4\times4}$ and $\mathcal{C} \in \boldsymbol{R}^{4\times4}$ denote the density matrix and the dissipative matrix given by

$$\mathcal{A} = \begin{bmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & \rho_{11} & 0 & \rho_{12} \\ \rho_{12} & 0 & \rho_{22} & 0 \\ 0 & \rho_{12} & 0 & \rho_{22} \end{bmatrix},\ \mathcal{C} = b(x) \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

where $\rho_{11} = \rho_1 - \rho_{12}$, $\rho_{22} = \rho_2 - \rho_{12}$, $\rho_1 = \rho_1(x)$ (respectively, $\rho_2 = \rho_2(x)$) is the mass of solid (respectively, fluid) per unit of the aggregate, and $\rho_{12} = \rho_{12}(x)$ is a mass coupling parameter between fluid and solid; $b = b(x)$ is the dissipation coefficient for Ω.

From physical consideration, it will be assumed that

$$\rho_{11}\rho_{22} - \rho_{12}^2 > 0,\quad x \in \bar{\Omega}, \tag{1.2}$$

$$0 < b_* \le b(x) \le b^* < \infty,\quad x \in \bar{\Omega}. \tag{1.3}$$

Then, it follows that $\mathcal{A}$ is positive-definite and $\mathcal{C}$ is nonnegative.

We shall impose initial conditions

$$u(x,0) = u^0,\ x \in \Omega,\quad \frac{\partial u}{\partial t}(x,0) = v^0,\ x \in \Omega, \tag{1.4}$$

and the homogeneous boundary conditions

$$(\theta_1(u) \cdot n, \theta_2(u) \cdot n, s(u)) = 0,\quad (x,t) \in \partial\Omega \times [0,T], \tag{1.5}$$

where $n = n(x)$ is the outward unit normal along $\partial\Omega$.

In this paper we shall analyze superconvergence phenomena for the numerical solution of (1.1). The analysis is based on a regularity assumption on the solution of (1.1) [6], [7], and the general method of an asymptotic expansion, called a quasi-projection, of the approximate solution [4]. We shall also rely on some earlier results on the subject [6], [7], which analyzed the existence and uniqueness of solution of (1.1) and the Galerkin procedure for the approximate solution of such equations.