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SECOND-ORDER METHODS FOR SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

In this paper we discuss the numerical methods with second-order accuracy for solving stochastic differential equations. An unbiased sample approximation method for $I_n = \int_{t_n}^{t_{n+1}} (B_u - B_{t_n})^2 du$ is proposed, where $\{B_u\}$ is a Brownian motion. Then second-order schemes are derived both for scalar cases and for system cases. The errors are measured in the mean square sense. Several numerical examples are included, and numerical results indicate that second-order schemes compare favorably with Euler's schemes and 1.5th-order schemes.

§1. Introduction

In this paper we discuss an approach of numerical solution for stochastic differential equations (abbreviated SDE) with second-order accuracy.

Assume that $\underline{B}_t$ is an m–dimensional Brownian motion on $(\Omega, \Im, P)$, and $\Im_t = \sigma(\underline{B}_s, s \le t)$ is an increasing family of sub–sigma-algebra of $\Im$.

Consider a SDE on $(\Omega, \Im_t, \Im, P)$, as in [1] or [15]:

$$dX(t) = b(X(t), t)dt + \sigma(X(t), t)dB_t, \quad \underline{X}(0) = X_0, \tag{1.1}$$

where $\underline{b}$ and σ are two sufficiently smooth functions satisfying the Lipshitz condition with respect to t, x. For simplicity, we only consider σ independent of t and $\underline{x}$, but the proof given here is valid for the general case without any more essential difficulties.

SDE (1.1) has exerted a profound impact on the modeling and analysis of problems in physics [2], chemistry [3], biology [4] and other fields [5, 6]. So more and more authors pay their attention to the numerical method for solving the SDE, such as H.J. Kushner, J.M.C. Clark, C.C. Chang [7,8,9,10,16]. Since solutions of the PDE can be expressed as functionals of solutions of the SDE, numerical solutions of the SDE can also be applied

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to numerical calculations of solutions of the PDE. H.J. Kushner, N.J. Rao succeeded in obtaining numerical solutions of the PDE by computing functionals of the numerical solutions of the SDE [7,12]. However, the numerical methods of highest order accuracy for solving the SDE used by the authors above are of order 1.5 (a numerical solution X_n^t of equation (1.1) is said to be of order α if

$$E\left(\frac{\sum_{i=1}^N \psi(X_n^i)}{N} - E\psi(X(t_n))\right)^2 \le \frac{C_1}{N} + C_2 h^{2\alpha},$$

where $X(t_n)$ is the solution of equation (1.1); see [16] or Section 3.). In order to get higher order accuracy, as pointed out by C.C.Chang and W.Rumelin [11,13], the main difficulty is how to simulate $I_n = \int_{t_n}^{t_{n+1}} (B_u - B_{t_n})^2 du$, where B_u is a one-dimensional Brownian motion. Because of the complexity of the distribution of I_n [14], it seems difficult to directly sample I_n. In this paper an unbiased sample approximation for I_n is proposed. Then second-order accuracy numerical methods both for the scalar case and the system case are drived. Numerical results also show that the second-order method produces errors smaller than the 1.5th-order method does.

§2. An Unbiased Sample Approximation for I_n

First, we outline the probability background for later use.

Definition 2.1 (martingale). *Suppose that the real valued stochastic process $Y(t)$ defined on $(\Omega, \Im, P)$ is adaptable to $\{\Im_t\}$ which is an increasing family of sub-sigma-algebra of $t \ge 0$. $\{Y(t), \Im_t, +\infty > t \ge 0\}$ is called a martingale, if $\forall t \ge 0$, $s \ge 0$, with probability one*

$$E|Y(t)| < +\infty, \qquad E(Y(t+s)|\Im_t) = Y(t).$$

Itô Formula. Let

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

and $F(x,t)$ be a continuous function on $R^n \times R^1$ together with $F(\cdot, t) \in C^1$ and $F(x, \cdot) \in C^2$. Then $F(x(t), t)$ satisfies

$$dF(X_t, t) = F_x(X_t, t)dX_t + F_t(X_t, t)dt + \frac{1}{2}F_{xx}(X_t, t)\sigma^2 dt.$$

For example, when $F(x) = X^2, dx(t) = dB_t$,

$$B_t^2 - B_s^2 = 2\int_s^t B_u dB_u + (t-s), \quad t > s > 0. \tag{2.1}$$

As stated earlier, to sample I_n directly seems difficult. Nevertheless, if we have a sufficiently good approximation of I_n which can be easily sampled, then the second-order scheme can still be obtained. Proposition 2.1 suggests a method for this sample approximation of I_n.

Suppose that

$$\Pi = (t_0 = 0, \ldots, t_{i+1} = t_i + h, \ldots, t_N = T)$$